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ON EQUAL SUMS OF LIKE POWERS

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I

Let $A_{i\,(=1,\,2,\,3,\,4)}$ and B_i be eight integers such that

$$\sum_{i=1}^{4} A_i^n = \sum_{i=1}^{4} B_i^n \qquad (n = 1, 3, 5). \tag{1}$$

Then, if we put

$$\begin{aligned} &A_1 = x - y,\ A_2 = y + w,\ A_3 = z - v,\ A_4 = v + u, \\ &B_1 = x + y,\ B_2 = w - y,\ B_3 = z + v,\ B_4 = u - v, \end{aligned} \tag{2}$$

we have to determine x, y, z, u, v and w so as to satisfy the Diophantine system

$$\begin{aligned} v\,(u^2 - x^2) &= y\,(x^2 - w^2)\,(2v)^2 + (u + z)^2 + (u - z)^2 \\ &= (2y)^2 + (x + w)^2 + (x - w)^2. \end{aligned} \tag{3}$$

Now, by means of Goldbach's identity

$$\begin{aligned} p^2 + q^2 - (3r - p - q)^3 &= (2r - p)^2 + (2r - q)^2 \\ &\quad + (r - p - q)^2, \end{aligned} \tag{4}$$

we see that Eq. (3_2) is satisfied if we have

$$\begin{aligned} &v = \tfrac{1}{2}p,\ u + z = 3r - p - q,\ u - z = q \\ &y = \tfrac{1}{2}(2r - p),\ \ x + w = 2r - q,\ \ x - w = r - p - q. \end{aligned} \tag{5}$$

Hence we get

$$\begin{aligned} u &= \tfrac{1}{2}(3r - p) \\ z = x &= \tfrac{1}{2}(3r - p - 2q) \\ w &= \tfrac{1}{2}(r + p) \end{aligned}$$

and the first of Eq. (3) takes the form

$$2r^2 - 3\,(p + q)\,r + (p^2 + pq + q^2) = 0. \tag{6}$$

Therefore,

$$r = \frac{3(p+q) \pm \sqrt{p^2 + 10pq + q^2}}{4}$$

and, to exclude the case $r =$ irrational, it is necessary and sufficient to take

$$p^2 + 10pq + q^2 = t^2.$$

From this we find

$$p = 2ab + 10b^2,\ q = a^2 - b^2,\ t = a^2 + 10ab + b^2$$

and then we have

$$r_1 = a^2 + 4ab + 7b^2, \qquad r_2 = \tfrac{1}{2}(a^2 - 2ab + 13b^2).$$

Substituting the foregoing values in the Eqs. (5_1), (5_4) and (6) we get

$x_1 = z_1 = \frac{1}{2}(a^2 + 10ab + 13b^2)$	$x_2 = z_2 = \frac{1}{4}(-a^2 - 10ab + 23b^2)$
$y_1 = a^2 + 3ab + 2b^2$	$y_2 = \frac{1}{2}(a^2 - 4ab + 3b^2)$
$w_1 = \frac{1}{2}(a^2 + 6ab + 17b^2)$	$w_2 = \frac{1}{2}(a^2 + 2ab + 33b^2)$
$v_1 = ab + 5b^2$	$v_2 = ab + 5b^2$
$u_1 = \frac{1}{2}(3a^2 + 10ab + 11b^2)$	$u_2 = \frac{1}{4}(3a^2 - 10ab + 19b^2)$.

Thus, from (2) and (1), we see that either

$$-a^2 + 4ab + 9b^2,\ 3a^2 + 12ab + 21b^2,\ a^2 + 8ab + 3b^2,\ 3a^2 + 12ab + 21b^2$$

$$\overset{n}{=} 3a^2 + 16ab + 17b^2,\ -a^2 + 13b^2,\ a^2 + 12ab + 23b^2,\ 3a^2 + 8ab + b^2,\ (\text{where } n = 1,\ 3,\ 5), \qquad (7^*)$$

or

$$-3a^2 - 2ab + 17b^2,\ 3a^2 - 6ab + 39b^2,\ -a^2 - 14ab + 3b^2,\ 3a^2 - 6ab + 39b^2$$

$$\overset{n}{=} a^2 - 18ab + 29b^2,\ -a^2 + 10ab + 27b^2,\ -a^2 - 6ab + 43b^2,\ 3a^2 - 14ab - b^2 \quad (n = 1,\ 3,\ 5). \qquad (8)$$

* The separation of two sets of numbers by the symbol $\overset{n}{=}$ denotes that they have the same sums of kth power for $k = 1, 2, \ldots n$, or for noted values of n.

Hence we have a process for finding solutions of the system (1) and in the case: $A_4 = A_2$, $A_1 + A_2 = B_1 + B_2$.

Also, by means of (3), we find that either

$$\left.\begin{aligned}&(2ab + 10b^2)(2a^2 + 10ab + 12b^2)(a^2 - b^2)\\&\quad = (2a^2 + 6ab + 4b^2)(a^2 + 8ab + 15b^2)\\&\qquad \times (2ab - 2b^2)\\&(2ab + 10b^2)^2 + (2a^2 + 10ab + 12b^2)^2 + (a^2 - b^2)^2\\&\quad = (2a^2 + 6ab + 4b^2)^2 + (a^2 + 8ab + 15b^2)^2\\&\qquad + (2ab - 2b^2)^2\end{aligned}\right\} \tag{9}$$

or

$$\left.\begin{aligned}&(2ab + 10b^2)\left(\frac{a^2 - 10ab + 21b^2}{2}\right)(a^2 - b^2)\\&\quad = (a^2 + 4ab + 3b^2)(-2ab + 14b^2)\\&\qquad \times \left(\frac{-a^2 - 6ab - 5b^2}{2}\right)\\&(2ab + 10b^2)^2 + \left(\frac{a^2 - 10ab + 21b^2}{2}\right)^2 + (a^2 - b^2)^2\\&\quad = (a^2 - 4ab + 3b^2)^2 + (-2ab + 14b^2)^2\\&\qquad + \left(\frac{-a^2 - 6ab - 5b^2}{2}\right)^2.\end{aligned}\right\} \tag{10}$$

It is easy to see that the equalities (9) express the condition that two rectangular parallelopipeds shall have integral edges, equal volumes and equal diagonals, if all the factors are positive.

Though the linear substitutions $a = a_1 - 5b_1$, $b = 2b_1$ transform, finally, the relation (7) into (8), however we consider them both, in order to find some other results. Also we consider (9) and (10).

II

From (4) we get the double relation

$$p,\ q,\ 3r - p - q \overset{2}{=} 2r - p,\ 2r - q,\ p + q - r.$$

Hence (9_2) and (10_2) become

$$2ab + 10b^2,\ 2a^2 + 10ab + 12b^2,\ a^2 - b^2 \overset{2}{=} 2a^2 + 6ab + 4b^2,\ a^2 + 8ab + 15b^2,\ 2b^2 - 2ab$$

$$a^2 - 4ab + 3b^2,\ -2ab + 14b^2,\ \frac{a^2 + 6ab + 5b^2}{2}$$

$$\overset{2}{=} 2ab + 10b^2,\ \frac{a^2 - 10ab + 21b^2}{2},\ a^2 - b^2,$$

from which we find

$$2a^2+10ab+12b^2,\ a^2-4ab+3b^2,\ -2ab+14b^2,\ \frac{a^2+6ab+5b^2}{2}$$

$$\overset{2}{=} 2a^2 + 6ab + 4b^2,\ a^2 + 8ab + 15b^2,\ -2ab + 2b^2,$$

$$\frac{a^2 - 10ab + 21b^2}{2}.$$

On the other hand, from (9_1) and (10_1), we take

$$(2a^2+10ab+12b^2)\ (a^2-4ab+3b^2)\ (-2ab+14b^2)\ \frac{(a^2+6ab+5b^2)}{2}$$

$$= (2a^2 + 6ab + 4b^2)\,(a^2 + 8ab + 15b^2)\,(-2ab + 2b^2)$$

$$\times \left(\frac{a^2 - 10ab + 21b^2}{2}\right).$$

Therefore we have a process for finding solutions of the system

$$\mathrm{A}_1,\ \mathrm{A}_2,\ \mathrm{A}_3,\ \mathrm{A}_4 \overset{2}{=} \mathrm{B}_1,\ \mathrm{B}_2,\ \mathrm{B}_3,\ \mathrm{B}_4$$

$$\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\mathrm{A}_4 = \mathrm{B}_1\mathrm{B}_2\mathrm{B}_3\mathrm{B}_4.$$

Thus,

$$4,\ 8,\ 30,\ 35 \overset{2}{=} 3,\ 14,\ 20,\ 40$$

$$4.8.30.35 = 3.14.20.40$$

and the polynomials

$$x^4 - 77x^3 + 1862x^2 - 14680x + 33600,$$

$$x^4 + 77x^3 + 1862x^2 - 16120x + 33600$$

have respectively the roots (4, 8, 30, 35) and (3, 14, 20, 40).

III

A two-parameter solution of the system

$$x_1 + x_2 + x_3 + x_4 = z_1 + z_2 + z_3 + z_4$$

$$x_1 + x_4 = x_2 + x_3 = z_1 + z_4 = z_2 + z_3 \qquad (11)$$

$$x_1x_2x_3x_4 = z_1z_2z_3z_4$$

may be found from (7).

In fact, for $A_1 + A_2 = B_1 + B_2$, $A_3 + A_4 = B_3 + B_4$, we find that every solution of (1) satisfies the system

$$(A_1 + A_3) + (A_1 + A_4) + (A_2 + A_3) + (A_2 + A_4)$$
$$= (B_1 + B_3) + (B_1 + B_4) + (B_2 + B_3) + (B_2 + B_4)$$

$$(A_1 + A_3)(A_1 + A_4)(A_2 + A_3)(A_2 + A_4)$$
$$= (B_1 + B_3)(B_1 + B_4)(B_2 + B_3)(B_2 + B_4).$$

Therefore, from (7), we get the following identities

$$(6ab + 6b^2) + (a^2 + 8ab + 15b^2) + (2a^2 + 10ab + 12b^2) + (3a^2 + 12ab + 21b^2)$$
$$= (2a^2 + 14ab + 20b^2) + (3a^2 + 12ab + 9b^2) + (6ab + 18b^2) + (a^2 + 4ab + 7b^2)$$

and

$$(6ab + 6b^2)(a^2 + 8ab + 15b^2)(2a^2 + 10ab + 12b^2) \times (3a^2 + 12ab + 21b^2)$$
$$= (2a^2 + 14ab + 20b^2)(3a^2 + 12ab + 9b^2)(6ab + 18b^2) \times (a^2 + 4ab + 7b^2),$$

by means of which we take particular solutions of the system (11).

Thus,

$$6 + 12 + 15 + 21 = 7 + 9 + 18 + 20$$
$$6 + 21 = 12 + 15 = 7 + 20 = 9 + 18.$$
$$6.12.15.21 = 7.9.18.20.$$

IV

The relation (8), for $a = -a_0 + 4b_0$, $b = a_0$, may be finally written

$$a_0^2 + a_0b_0 - 3b_0^2,\ 3(a_0^2 - a_0b_0 + b_0^2),\ a_0^2 - 3a_0b_0 - b_0^2, \times 3(a_0^2 - a_0b_0 + b_0^2)$$
$$\doteq 3a_0^2 - 5a_0b_0 + b_0^2, a_0^2 + 3a_0b_0 - b_0^2, 3a_0^2 - a_0b_0 - b_0^2, a_0^2 - 5a_0b_0 + 3b_0^2, \quad (n = 1, 3, 5)$$

whilst (7), for $a = a_0 + b_0$, $b = a_0 - b_0$, gives

$$3a_0^2 - 5a_0b_0 + b_0^2,\ 3(3a_0^2 - 3a_0b_0 + b_0^2),\ 3a_0^2 - a_0b_0 - b_0^2, 3(3a_0^2 - 3a_0b_0 + b_0^2)$$

$$\overset{n}{=} 9a_0^2 - 7a_0b_0 + b_0^2, 3a_0^2 - 7a_0b_0 + 3b_0^2, 9a_0^2 - 11a_0b_0 + 3b_0^2, 3a_0^2 + a_0b_0 - b_0^2 \qquad (n = 1, 3, 5).$$

Adding the foregoing two relations, we take, after the omission of the index 0, the new relation

$$a^2 - 3ab - b^2, \quad a^2 + ab - 3b^2, \quad 3(a^2 - ab + b^2), \quad 3(a^2 - ab + b^2), \; 3(3a^2 - 3ab + b^2), \; 3(3a^2 - 3ab + b^2)$$

$$\overset{n}{=} a^2 + 3ab - b^2, a^2 - 5ab + 3b^2, 3a^2 - 7ab + 3b^2, 3a^2 + ab - b^2, 9a^2 - 11ab + 3b^2, 9a^2 - 7ab + b^2 \qquad (n = 1, 3, 5), \qquad (12)$$

by means of which we get particular solutions of the system

$$A_1, A_2, A_3, A_3, A_4, A_4 \overset{n}{=} B_1, B_2, B_3, B_4, B_5, B_6 \qquad (n = 1, 3, 5).$$

Thus, for $a = 2$, $b = -1$, we have

$$9, (-1), 21, 21, 57, 57 \overset{n}{=} -3, 17, 29, 9, 51, 61, \qquad (n = 1, 3, 5)$$

that is

$$3, 21, 21, 57, 57 \overset{n}{=} 1, 17, 29, 51, 61. \qquad (n = 1, 3, 5)$$

V

By interchanging the role of a and b we have, from (12),

$$-a^2 + 3ab + b^2, \; 3a^2 - 5ab + b^2, \; 3a^2 - 7ab + 3b^2, \; -a^2 + ab + 3b^2, \; 3a^2 - 11ab + 9b^2, \; a^2 - 7ab + 9b^2$$

$$\overset{n}{=} -a^2 - 3ab + b^2, -3a^2 + ab + b^2, 3(a^2 - ab + b^2), 3(a^2 - ab + b^2), 3(a^2 - 3ab + 3b^2), 3(a^2 - 3ab + 3b^2) \qquad (n = 1, 3, 5)$$

Adding the above relation to (12) we take

$$3(3a^2 - 3ab + b^2), \; 3(3a^2 - 3ab + b^2), \; a^2 + ab - 3b^2, \; a^2 - 7ab + 9b^2, \; 3a^2 - 11ab + 9b^2, \; -a^2 + ab + 3b^2, \; 3a^2 - 5ab + b^2$$

$$\overset{n}{=} 3(a^2 - 3ab + 3b^2),\ 3(a^2 - 3ab + 3b^2),\ -3a^2+ab+b^2,$$
$$9a^2 - 7ab + b^2,\ 9a^2 - 11ab + 3b^2,\ 3a^2 + ab - b^2,$$
$$a^2 - 5ab + 3b^2 \qquad (n = 1, 3, 5).$$

This affords a solution of the system

$$A, A, A_1, A_2, \ldots\ldots, A_5 \overset{n}{=} B, B, B_1, B_2, \ldots\ldots\ldots, B_5 \ (n = 1, 3, 5),$$

depending upon two parameters.

VI

In order to obtain solutions of the system

$$A_1, A_2, A_3 \overset{n}{=} B_1, B_2, B_3, B_4 \qquad (n = 1, 3, 5) \qquad (13)$$

we may proceed as follows:

Starting with the system

$$M_1, M_2, M_3, M_4 \overset{3}{=} N_1, N_2, N_3, N_4 \qquad (14)$$

and putting

$$\tfrac{1}{4}(M_1 + M_2 + M_3 + M_4) = H$$

we may write the relation

$$M_1 - H, M_2 - H, M_3 - H, M_4 - H$$
$$\overset{m}{=} N_1 - H, N_2 - H, N_3 - H, N_4 - H,$$
$$(m = 1, 2, 3, 5), \qquad (15)$$

where the sum of the first powers of the terms of each member is equal to zero.

Now, if

$$H = N_2,$$

we get

$$N_2 - N_1, M_3 - N_2, M_4 - N_2$$
$$\overset{n}{=} N_3 - N_2, N_4 - N_2, N_2 - M_2, N_2 - M_1,$$
$$(n = 1, 3, 5).$$

If one applies this method in the particular relation

$$2,\ 45,\ 76,\ 85 \overset{3}{=} 1,\ 52,\ 65,\ 90,$$

in which

$$\tfrac{1}{4}(2 + 45 + 76 + 85) = 52 = N_2,$$

one finds that a solution of the system (13) is afforded by

$$52 - 1,\ 76 - 52,\ 85 - 52 \overset{n}{=} 65 - 52,\ 90 - 52,\ 52 - 45,\ 52 - 2, \qquad (n = 1, 3, 5)$$

that is

$$51,\ 24,\ 33 \overset{n}{=} 13,\ 38,\ 7,\ 50, \qquad (n = 1, 3, 5)$$

where

$$51 = 13 + 38.$$

Then we shall confine our attention to the system (14), under the restriction

$$N_2 = \tfrac{1}{4}(M_1 + M_2 + M_3 + M_4).$$

For this, we consider the system

$$2v,\ z + x - y,\ 2z - 2(v + x),\ z + x + y \overset{3}{=} v,\ z,\ z + x,\ 2z - (v + x),$$

in order to find a solution of (14), in which the supplementary conditions

$$M_1 = 2N_1,\ \ M_2 + M_4 = 2N_3$$

exist, as in the case of the above example.

From the system under consideration we get the equivalent system

$$3v^2 + 3vx + 2x^2 - z(2v + x) + y^2 = 0$$

$$zx(7x - 3z + 12v) + 2zv(3v - 2z) - 7vx(v + x) - 2x^3 + 2(z + x)y^2 = 0,$$

from which, eliminating y^2, we find that

$$z^2 + 6x^2 + 13v^2 - 5zx - 10zv + 13xv = 0.$$

Starting from the particular solution $z = 2,\ x = -1,\ v = 1$ of the last equation, we put $z = 2r + u,\ x = -r + w,\ v = r$. Then the above equation gives r as a rational function of u, w, so that

$$z = 3u^2 - uw + 12w^2$$
$$x = -u^2 + 6uw + 3w^2$$
$$v = u^2 - 5uw + 6w^2$$

and the first equation of the equivalent system becomes

$$y^2 = u^4 + 8u^3w - 11u^2w^2 - 18uw^2. \tag{16}$$

Therefore (14) may be written

$$\begin{aligned} &2u^2 - 10uw + 12w^2,\ (2u^2 + 5uw + 15w^2) - y,\\ &\qquad 6u^2 - 4uw + 6w^2,\ (2u^2 + 5uw + 15w^2) + y\\ &\overset{3}{=} u^2 - 5uw + 6w^2,\ 3u^2 - uw + 12w^2,\ 2u^2 + 5uw + 15w^2\\ &\qquad 6u^2 - 3uw + 15w^2 \end{aligned}$$

where the integers u, w, y satisfy the Eq. (16).

Methods for resolving such equations as (16) have been given by several writers.

Hence, we conclude as follows:

$\ll$ Particular solutions of the system (13) are given by means of the equalities

$$\begin{aligned} &2u^2 + 4uw + 6w^2,\ 3u^2 - 3uw - 6w^2,\ (-u^2 + 6uw + 3w^2) + y\\ &\overset{n}{=} -u^2 + 6uw + 3w^2,\ 3u^2 - 2uw + 3w^2,\ u^2 + 9uw,\\ &\qquad (w^2 - 6uw - 3w^2) + y, \qquad\qquad (n = 1,\ 3,\ 5) \end{aligned}$$

where the integers u, w, y satisfy the equation (16), that is to say

$$y^2 = u(u - 2w)(u + w)(u + 9w).$$

Also, from (15) and under the above restriction, we get

$$\begin{aligned} &-(u^2 + 9uw),\ -u^2 + 6uw + 3w^2 - y,\ 3u^2 - 3uw - 6w^2,\\ &\qquad -u^2 + 6uw + 3w^2 + y\\ &\overset{m}{=} -(2u^2 + 4uw + 6w^2),\ -u^2 + 6uw + 3w^2,\\ &\qquad 3u^2 - 2uw + 3w^2, \end{aligned}$$

where

$$m = 1,\ 2,\ 3,\ 5 \gg$$

Thus, for

$$u = 2,\ \ w = -3,\ \ y = 20,$$

we get

$$38,\ -24,\ 7 \overset{n}{=} -13,\ 51,\ -50,\ 33,$$

or

$$24,\ 33,\ 51 \overset{n}{=} 7,\ 13,\ 38,\ 50 \qquad (n = 1,\ 3,\ 5)$$

and

$$50, (-33), (-24), 7 \overset{m}{=} -38, (-13), 51 \qquad (m = 1, 2, 3, 5)$$

For $u = 81,\ w = 40,\ y = 2079$ we get finally

$$5893,\ 6001,\ 11907 \quad 121,\ 5200,\ 6586,\ 11894, \qquad (n = 1,\ 3,\ 5) \text{ etc.}$$

Here we remark that by means of the relations

$$R_1,\ R_2,\ R_3 \overset{m}{=} \mathcal{U}_1,\ \mathcal{U}_2,\ \mathcal{U}_3,\ \mathcal{U}_4 \qquad (m = 1, 2, 3, 5)$$
$$-R_1 = R_2 + R_3,$$

we get

$$(R_1 + 0)^2 + (R_1 + R_2)^2 + (R_1 + R_3)^2$$
$$= (\mathcal{U}_1 + \mathcal{U}_2)^2 + (\mathcal{U}_1 + \mathcal{U}_3)^2 + (\mathcal{U}_1 + \mathcal{U}_4)^2$$
$$(R_1 + 0)(R_1 + R_2)\ (R_1 + R_3) = (\mathcal{U}_1+\mathcal{U}_2)\ (\mathcal{U}_1+\mathcal{U}_3)(\mathcal{U}_1+\mathcal{U}_4).$$

Thus, from

$$51, (-13),\ (-38) \overset{n}{=} 50,\ (-33),\ (-24),\ 7,$$

we have

$$51^2 + 38^2 + 13^2 = 17^2 + 26^2 + 57^2$$
$$51.38.13 = 17.26.57.$$

And, to finish, we note that a two-parameter solution of (13) is afforded by the values

$$A_1 = a^5 + 2a^4b - 2a^3b^2 - 5a^2b^3 - 2ab^4$$

$$A_2 = a^4b + 3a^3b^2 - 3ab^4 - b^5$$

$$A_3 = a^5 + a^4b - a^3b^2 + a^2b^3 + 3ab^4 + b^5$$

$$B_1 = a^5 + a^4b - 2a^3b^2 - 4a^2b^3 - 4ab^4 - b^5$$

$$B_2 = 2a^4b + 3a^3b^2 - a^2b^3 - ab^4$$

$$B_3 = -a^4b + 5a^2b^3 + 4ab^4 + b^5$$

$$B_4 = a^5 + 2a^4b - a^3b^2 - 4a^2b^3 - ab^4$$

(where $A_3 = B_3 + B_4$) a and b being arbitrary integers.

STUDIES IN THE FORMATION OF HETEROCYCLIC RINGS CONTAINING NITROGEN

Part VI. Condensation of 4-Chloro-*o*-Phenylenediamine with Aromatic Aldehydes

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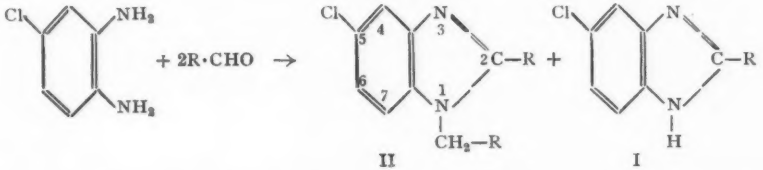
The condensation of 4-chloro-*o*-phenylenediamine with benzaldehyde and *p*-nitrobenzaldehyde under different conditions was studied by Fischer and Limmer[1] who reported the formation of dianils, which could be converted into the corresponding 1 : 2-disubstituted benziminazoles. The position of the chloro group (5 or 6) in the benziminazoles was not ascertained by these workers. In a recent publication,[2] the present authors established by unambiguous synthetic methods that the chloro group occupied position 5.

A systematic study of the condensation between 4-chloro-*o*-phenylenediamine and two moles of seventeen aromatic aldehydes (other than benzaldehyde[2]) containing alkyl, hydroxy, alkoxy, halo, nitro and amino substituents has been made in the present investigation with a view to finding out the influence of substituents in the aldehydes as well as of the chloro group of the diamine on the nature and relative yields of the reaction products. The condensations have been carried out following the modified Hinsberg's procedure,[3] and the results obtained are summarised in Table I.

The reaction has been found to yield mono (I) and di (II) substituted benziminazoles as in the other condensations.[3, 4] On account of the presence of low-melting resinous materials formed in the condensations, much difficulty has been encountered in processing the reaction products.



It was shown previously in the case of condensations with *o*-phenylenediamine[3] and 4-methyl-*o*-phenylenediamine,[4] that mono and di anils are possible intermediates in the formation of mono and disubstituted benziminazoles. It was also shown that electron-releasing groups like hydroxy

TABLE I

Products of condensation of 4-chloro-o-phenylenediamine with R.CHO

No.	R-	% yield of (II)	% yield of (I)
1.	*p*-tolyl	..	2
2.	*o*-hydroxyphenyl	66	25
3.	*m*-hydroxyphenyl	95	..
4.	*p*-hydroxyphenyl	95	..
5.	*m*-methoxy-*p*-hydroxyphenyl	95	..
6.	3:4-methylenedioxyphenyl..	73	13
7.	*p*-methoxyphenyl	78	15
8.	*o*-chlorophenyl	..	45
9.	*p*-chlorophenyl	30	4
10.	2:4-dichlorophenyl	..	42
11.	*p*-bromophenyl	31	..
12.	*o*-nitrophenyl	..	50
13.	*m*-nitrophenyl	87	..
14.	*p*-nitrophenyl	68*	13
15.	2-hydroxy-5-nitrophenyl†	..	..
16.	*p*-dimethylaminophenyl	87	..
17.	1-naphthyl	88	..

* Compound reported in literature. † Dianil isolated in 95% yield.

and amino present in the aldehydes facilitate the exclusive formation of disubstituted benziminazoles, while electron-withdrawing groups like halo and nitro favour the formation of considerable amounts of mono derivatives. The results presented in this paper further corroborate these views. The isolation of a stable dianil in the reaction with 5-nitrosalicylaldehyde which could be converted to the corresponding benziminazole by refluxing

in acetic acid, lends support to the intermediary nature of dianils in the formation of disubstituted benziminazoles.

The influence of substituents in the aldehydes on the course of the reaction seems to be affected to a certain extent by the chloro group of the diamine. The condensation of the diamine with salicylaldehyde yielded, in addition to the 1:2-disubstituted benziminazole, a considerable amount of the mono derivative. *o*-Chloro- and *o*-nitro- benzaldehydes afforded exclusively monosubstituted benziminazoles in their reaction with this diamine whereas with other diamines[3, 4] these aldehydes gave appreciable amounts of disubstituted benziminazoles also. These results indicate that the chloro group present in the diamine favours the formation of monosubstituted benziminazoles. *p*-Tolualdehyde gave a very small amount of the mono derivative and large amounts of high-melting products. Such high-melting and indefinite products have also been met with in the reactions of the diamine with halo and alkoxy aldehydes, on account of which the yields of benziminazoles have been rather low.

EXPERIMENTAL

All m.p.'s are uncorrected. The micro-analyses were carried out by one of the authors (C. V. R.). Unless otherwise mentioned, ethyl alcohol is the solvent used for recrystallisation.

General procedure for the condensations.—4-Chloro-*o*-phenylenediamine (one mole) and aromatic aldehyde (two moles) were dissolved separately in a slight excess of glacial acetic acid, and mixed together in a conical flask, when evolution of heat was noticed. The reaction was allowed to proceed for one hour with occasional stirring. If any crystalline solid separated by this time, it was filtered and the clear filtrate was added to excess of crushed ice with vigorous stirring. The compound that separated was filtered, and the filtrate was made ammoniacal, when a small amount of another compound was obtained. The compounds thus isolated were purified and characterised.

2-(p-*Methylphenyl*)-5-(*or* 6)-*chloro benziminazole* was obtained by condensation with *p*-tolualdehyde, from the ammoniacal solution. It crystallised in prismatic rods, m.p. 228° (Found: C, 68·8; H, 5·0; N, 12·1; $C_{14}H_{11}N_2Cl$ requires C, 69·3; H, 4·5; N, 11·5%).

1-(o-*Hydroxybenzyl*)-2-(o-*hydroxyphenyl*)-5-*chloro benziminazole* was obtained from the dilute acid solution of the salicylaldehyde condensation. The compound came out as needles from alcohol-acetone, m.p. 187° (Found:

C, 68·1; H, 4·8; N, 8·0; $C_{20}H_{15}N_2O_2Cl$ requires C, 68·5; H, 4·3; N, 8·0%).

2-(o-*Hydroxyphenyl*)-5-(*or* 6)-*chloro benziminazole* was isolated from the ammoniacal solution of the above condensation. On purification, it came out as tiny rods, m.p. 273° (Found: C, 63·5; H, 3·9; N, 11·8; $C_{13}H_9N_2OCl$ requires C, 63·8; H, 3·7; N, 11·5%).

1-(m-*Hydroxybenzyl*)-2-(m-*hydroxyphenyl*)-5-*chloro benziminazole* was obtained by condensation with *m*-hydroxybenzaldehyde mostly from the dilute acid solution and in small quantity from the ammoniacal solution. The product on recrystallisation gave tiny rectangular rods, m.p. 247° (Found: C, 68·0; H, 4·7; N, 8·4; $C_{20}H_{15}N_2O_2Cl$ requires C, 68·5; H, 4·3; N, 8·0%).

1-(p-*Hydroxybenzyl*)-2-(p-*hydroxyphenyl*)-5-*chloro benziminazole* was isolated from *p*-hydroxybenzaldehyde condensation, as in the above case. It crystallised in rectangular rods, m.p. 230° (Found: C, 68·3; H, 4·0; N, 8·0; $C_{20}H_{15}N_2O_2Cl$ requires C, 68·5; H, 4·3; N, 8·0%).

1-(m-*Methoxy*-p-*hydroxybenzyl*)-2-(m-*methoxy*-p-*hydroxyphenyl*)-5-*chloro-benziminazole* was prepared by condensation with vanillin and isolated as in the above two condensations. It was obtained as needles, m.p. 213° (Found: C, 64·0; H, 5·2; N, 7·0; $C_{22}H_{19}N_2O_4Cl$ requires C, 64·3; H, 4·6; N, 6·8%).

1-(3:4-*Methylenedioxybenzyl*)-2-(3:4-*methylenedioxyphenyl*)-5-*chloro benziminazole.*—In the condensation with piperonal, a sticky solid separated from the dilute acid solution, which on crushing with a little methanol, yielded a granular solid. The solid crystallised in rectangular rods, m.p 171° (Found: C, 65·6; H, 4·2; N, 7·0; $C_{22}H_{15}N_2O_4Cl$ requires C, 65·0; H, 3·7; N, 6·9%).

2-(3:4-*Methylenedioxyphenyl*)-5-(*or* 6)-*chloro benziminazole* was isolated from the ammoniacal solution of the above condensation and was obtained as leaflets, m.p. 205° (Found: C, 61·5; H, 3·8; N, 9·8; $C_{14}H_9N_2O_2Cl$ requires C, 61·7; H, 3·3; N, 10·3%).

1-(p-*Methoxybenzyl*)-2-(p-*methoxyphenyl*)-5-*chloro benziminazole* separated from the dilute acid solution of anisaldehyde condensation and was purified by recrystallisation, first from alcohol and then from petroleum ether, prismatic rods, m.p. 125° (Found: C, 69·2; H, 5·2; N, 8·0; $C_{22}H_{19}N_2O_2Cl$ requires C, 69·7; H, 5·0; N, 7·4%).

2-(p-*Methoxyphenyl*)-5-(*or* 6)-*chloro benziminazole* was obtained from the ammoniacal solution of the above condensation and on purification gave plates, m.p. 180° (Found: C, 64·7; H, 4·6; N, 10·9; $C_{14}H_{11}N_2OCl$ requires C, 65·0; H, 4·3; N, 10·8%).

2-(o-*Chlorophenyl*)-5-(*or* 6)-*chloro benziminazole*.—A sticky mass was obtained from the dilute acid solution of the condensation with *o*-chloro benzaldehyde, which on crushing with petroleum ether gave a granular solid. The solid crystallised in rectangular rods, m.p. 125° (Found: C, 59·0; H, 3·5; N, 10·6; $C_{13}H_8N_2Cl_2$ requires C, 59·3; H, 3·0; N, 10·6%).

1-(p-*chlorobenzyl*)-2-(p-*chlorophenyl*)-5-*chloro benziminazole* was obtained from the dilute acid solution of *p*-chlorobenzaldehyde condensation. The crude compound was purified by recrystallisation from alcohol and then from petroleum ether yielding needles, m.p. 150° (Found: C, 62·3; H, 3·7; N, 7·5; $C_{20}H_{13}N_2Cl_3$ requires C, 61·9; H, 3·4; N, 7·2%).

2-(p-*chlorophenyl*)-5-(*or* 6)-*chloro benziminazole* was isolated from the ammoniacal solution of the above condensation and on recrystallisation gave clusters of prismatic rods, m.p. 225° (Found: C, 58·8; H, 3·4; N, 10·5; $C_{13}H_8N_2Cl_2$ requires C, 59·3; H, 3·0; N, 10·6%).

2-(2: 4-*Dichlorophenyl*)-5-(*or* 6)-*chloro benziminazole* separated from the glacial acetic acid solution in the condensation with 2: 4-dichlorobenzaldehyde. The compound crystallised in tiny needles, m.p. 184° (Found: C, 52·1; H, 2·8; N, 9·5; $C_{13}H_7N_2Cl_3$ requires C, 52·4; H, 2·4; N, 9·4%).

1-(p-*Bromobenzyl*)-2-(p-*bromophenyl*)-5-*chloro benziminazole*.—In the *p*-bromobenzaldehyde condensation, a solid changing to a black resin separated out from the dilute acid solution. It was repeatedly crystallised from alcohol, and finally from petroleum ether yielding the pure benziminazole, prismatic rods, m.p. 155° (Found: C, 50·7; H, 3·2; N, 6·4; $C_{20}H_{13}N_2ClBr_2$ requires C, 50·4; H, 2·7; N, 5·9%).

2-(o-*Nitrophenyl*)-5-(*or* 6)-*chloro benziminazole* was obtained by the condensation with *o*-nitrobenzaldehyde, from the dilute acid solution. The compound was washed with a little benzene and recrystallised, shining yellow plates, m.p. 187° (Found: C, 56·5; H, 3·5; N, 15·5; $C_{13}H_8N_3O_2Cl$ requires C, 57·0; H, 2·9; N, 15·4%).

1-(m-*Nitrobenzyl*)-2-(m-*nitrophenyl*)-5-*chloro benziminazole* was isolated from the dilute acid solution in *m*-nitrobenzaldehyde condensation. It crystallised from benzene containing a small quantity of alcohol in

pale yellow needles, m.p. 193° (Found: C, 59·0; H, 3·5; N, 13·4; $C_{20}H_{13}N_4O_4Cl$ requires C, 58·8; H, 3·2; N, 13·7%).

2-(p-*Nitrophenyl*)-5-(*or* 6)-*chloro benziminazole* separated from the glacial acetic acid solution in the condensation with *p*-nitrobenzaldehyde. On purification, the compound came out as clusters of yellow needles, m.p. 257° (Found: C, 57·5; H, 3·3; N, 15·6; $C_{13}H_8N_3O_2Cl$ requires C, 57·0; H, 2·9; N, 15·4%).

1-(p-*Nitrobenzyl*)-2-(p-*nitrophenyl*)-5-*chloro benziminazole* was isolated from the dilute acid solution of the above condensation and on crystallisation yielded yellow prismatic rods, m.p. 227° (Fischer and Limmer[1] report m.p. 235°) (Found: C, 59·2; H, 3·7; N, 14·1; $C_{20}H_{13}N_4O_4Cl$ requires C, 58·8; H, 3·2; N, 13·7%).

N: *N'-Di*-(2-*hydroxy*-5-*nitrobenzylidene*)-4-*chloro*-o-*phenylenediamine* separated out from the glacial acetic acid solution in 5-nitrosalicylaldehyde condensation and crystallised from nitrobenzene in orange yellow needles, m.p. 260° (decomp.) (Found: C, 55·0; H, 3·2; N, 12·5; $C_{20}H_{13}N_4O_6Cl$ requires C, 54·5; H, 3·0; N, 12·7%).

1-(2-*Hydroxy*-5-*nitrobenzyl*)-2-(2-*hydroxy*-5-*nitrophenyl*)-5-*chloro benziminazole*, prepared by refluxing the above dianil in excess of acetic acid for twelve hours, crystallised from benzene in shining yellow prismatic rods, m.p. 314° (decomp.) (Found: C, 55·0; H, 3·5; N, 13·2; $C_{20}H_{13}N_4O_6Cl$ requires C, 54·5; H, 3·0; N, 12·7%).

1-(p-*Dimethylaminobenzyl*)-2-(p-*dimethylaminophenyl*)-5-*chloro benziminazole*.—The reaction between the diamine and *p*-dimethylaminobenzaldehyde was conducted at 100° as the reaction did not proceed satisfactorily at the laboratory tempreature. The benziminazole was isolated partly from dilute acid solution and mostly from the ammoniacal solution. It crystallised from benzene-acetone in shining pale yellow prismatic rods, m.p. 239° (Found: C, 71·1; H, 5·8; N, 14·5; $C_{24}H_{25}N_4Cl$ requires C, 71·2; H, 6·2; N, 13·9%).

1-(1-*Naphthyl methyl*)-2-(1-*naphthyl*)-5-*chloro benziminazole* was obtained by the condensation with 1-naphthaldehyde, from the dilute acid solution. The compound was washed with a little acetone and recrystallised, rectangular rods, m.p. 199° (Found: C, 80·5; H, 5·0; N, 6·8; $C_{28}H_{19}N_2Cl$ requires C, 80·3; H, 4·5; N, 6·7%).

SUMMARY

The condensation of 4-chloro-*o*-phenylenediamine with two moles of seventeen aromatic aldehydes under acidic conditions has been found to yield mono and disubstituted benziminazoles in varying proportions. The influence of substituents in the aldehydes on the mode of reaction and cyclisation has been found to be similar to that observed in the condensations with *o*-phenylenediamine.

REFERENCES

1. Fischer and Limmer .. *J. Prakt. Chem.*, 1906, **74** (2), 58.
2. Rao and Ratnam .. *Proc. Ind. Acad. Sci.*, 1958, **47**, 77.
3. ———— .. *Ibid.*, 1956, **43**, 173.
4. ———— .. *Ibid.*, 1957, **45**, 253.

SELF-RECIPROCAL FUNCTIONS IN THE FORM OF SERIES

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1. The object of this note is to find out a few Self-Reciprocal functions in the form of series. Following the usual notation, I denote a function $f(x)$ as R_μ, if it is Self-Reciprocal for Hankel Transforms of order μ so that it is given by.

$$f(x) = \int_0^\infty J_\mu(xy) f(y) \sqrt{xy}\, dy, \tag{1.1}$$

where $J_\mu(x)$ is a Bessel function of order μ. If $\mu = -\frac{1}{2}$, $f(x)$ is denoted as R_c while for $\mu = \frac{1}{2}$, $f(x)$ is written as R_s.

2. Dr. Willis[3] has obtained expansions for certain Integrals, in the form of infinite series. Further Dr. Brij Mohan[1] has obtained a class of kernels for Self-Reciprocal functions of Hankel Transforms. On applying these kernels in the Integrals given by Dr. Willis, we find that the expansions in series become Self-Reciprocal functions. Accordingly we show in the following lines that certain Self-Reciprocal functions can be represented as infinite series.

3. Dr. Willis[3] has shown that

$$\int_0^\infty f(x)\, e^{-mx} dx = \frac{1}{m}\left\{ f(0) + \frac{f^{(1)}(0)}{m} + \frac{f^{(2)}(0)}{m^2} + \ldots . \right\}. \tag{3.1}$$

Let $f(x)$ be R_c or R_s. Then according to Dr. Brij Mohan[1] the kernel

$$\bar{e}^x, \tag{3.2}$$

transforms

$$R_c\,(R_s) \text{ into } R_s\,(R_c)$$

Hence we find that

$$\int_0^\infty \bar{e}^{mx} f(x)\, dx, \tag{3.3}$$

is $R_s\,(R_c)$ according as $f(x)$ is $R_c\,(R_s)$.

Hence from (3.3) and (3.1), we conclude that

$$\frac{1}{x}\left\{f(0)+\frac{f^{(1)}(0)}{x}+\frac{f^{(2)}(0)}{x^2}+\ldots.\right\}. \tag{3.4}$$

is R_s (R_c) according as $f(x)$ is R_c (R_s).

4. Again, Dr. Willis[3] has shown that

$$\int_0^\infty f(y)\sin xy\,dy=\frac{1}{x}\left\{f(0)+\frac{f^{(1)}(0)}{x^2}+\frac{f^{(2)}(0)}{x^4}+\ldots.\right\}. \tag{4.1}$$

Let $f(x)$ be R_s. Then by the definition of an R_s function we find that

$$\int_0^\infty f(y)\sin xy\,dy=f(x). \tag{4.2}$$

Hence from (4.2) and (4.1), we conclude that

$$\frac{1}{x}\left\{f(0)+\frac{f^{(1)}(0)}{x^2}+\frac{f^{(2)}(0)}{x^4}+\ldots.\right\}, \tag{4.3}$$

is R_s if $f(x)$ is R_s.

5. Further, it has been shown by Dr. Willis[3] that

$$\int_0^\infty f(y)\,J_0(xy)\,dy=\frac{1}{x}\left\{f(0)-\frac{f^{(2)}(0)}{2\angle 1\,x^2}+\frac{1\cdot 3f^{(4)}(0)}{2^2\angle 2\,x^4}-\frac{1.3.5f^{(6)}(0)}{2^3\angle 3\,x^6}+\ldots.\right\}. \tag{5.1}$$

Let $f(x)$ be R_c or R_s.

Dr. Brij Mohan[1] has shown that the kernel

$$J_0(x), \tag{5.2}$$

transforms

$$R_c\,(R_s) \text{ into } R_s\,(R_c).$$

Hence from (5.1) and (5.2), we conclude that

$$\frac{1}{x}\left\{f(0)-\frac{f^{(2)}(0)}{2\angle 1\,x^2}+\frac{1\cdot 3f^{(4)}(0)}{2^2\angle 2\,x^4}-\frac{1.3.5.f^{(6)}(0)}{2^3\angle 3\,x^6}+\ldots.\right\}, \tag{5.3}$$

is R_s (R_c) according as $f(x)$ is R_c (R_s).

6. It has been further shown by Dr. Willis[3] that

$$\int_0^\infty f(x)\, J_1(xy)\, dy = \frac{f(0)}{x} + \frac{1}{x}\left\{\frac{f^{(1)}(0)}{x} + \frac{f^{(3)}(0)}{2 \angle 1\, x^3}\right.$$

$$\left. + \frac{1.3 f^{(5)}(0)}{2^2 \angle 2\, x^5} - \frac{1.3.5 f^{(7)}(0)}{2^3 \angle 3\, x^7} + \ldots\right\}. \tag{6.1}$$

Dr. Brij Mohan[1] has shown that the kernel

$$x^{\frac{1}{2}(\mu-\nu+1)}\, J_{\frac{1}{2}(\mu+\nu)}(x), \tag{6.2}$$

transforms R_μ into R_ν.

Putting $\nu = (\mu + 1)$ we find that the kernel

$$J_{(\mu+\frac{1}{2})}(x), \tag{6.3}$$

transforms R_μ into $R_{(\mu+1)}$; and in particular we find that

$$J_1(x), \tag{6.4}$$

transforms R_s into $R_{3/2}$.

Therefore, we find that

$$\int_0^\infty f(y)\, J_1(xy)\, dy, \tag{6.5}$$

is $R_{3/2}$ if $f(x)$ is R_s.

Hence from (6.5) and (6.1), we further conclude that

$$\frac{f(0)}{x} + \frac{1}{x}\left\{\frac{f^{(1)}(0)}{x} - \frac{f^{(3)}(0)}{2 \angle 1\, x^3} + \frac{1.3 f^{(5)}(0)}{2^2 \angle 2\, x^5}\right.$$

$$\left. - \frac{1.3.5 f^{(7)}(0)}{2^3 \angle 3\, x^7} + \ldots\right\}, \tag{6.6}$$

is $R_{3/2}$ if $f(x)$ is R_s.

7. Again Dr. Willis[3] has also shown that

$$\int_0^\infty f(y) \frac{\sin xy}{y}\, dy = \frac{\pi}{2} f(0) + \left\{\frac{f^{(1)}(0)}{x} - \frac{f^{(3)}(0)}{3x^3}\right.$$

$$\left. + \frac{f^{(5)}(0)}{5x^5} - \ldots\right\}, \tag{7.1}$$

so that

$$\int_0^\infty f(y)\frac{\sin xy}{xy}\,dy = \frac{1}{x}\left[\frac{\pi}{2}f(0) + \left\{\frac{f^{(1)}(0)}{x} - \frac{f^{(3)}(0)}{3x^3} + \frac{f^{5}(0)}{5x^5} - \ldots.\right\}\right]. \tag{7.2}$$

Dr. Brij Mohan[1] has shown that the kernel

$$\frac{\sin x}{x}, \tag{7.3}$$

transforms $R_c(R_{3/2})$ into $R_{3/2}(R_c)$.

Hence it follows from (7.2) and (7.3) that

$$\frac{1}{x}\left[\frac{\pi}{2}f(0) + \left\{\frac{f^{(1)}(0)}{x} - \frac{f^{(3)}(0)}{x^3} + \frac{f^{(5)}(0)}{5x^5} - \ldots.\right\}\right], \tag{7.4}$$

is $R_{3/2}(R_c)$ according as $R_c(R_{3/2})$.

8. It has been further shown by Dr. Willis[3] that

$$\int_0^\infty f(y)e^{y^2x^2}dy = \frac{\sqrt{\pi}}{2x}\left\{f(0) + \frac{1}{\angle 2}\cdot\frac{1}{2}\frac{f^{(2)}(0)}{x^2} + \frac{1}{\angle 4}\frac{1.3}{2^2}\frac{f^{(4)}(0)}{x^4} + \ldots.\right\} + \frac{2}{x}\left\{\frac{f^{(1)}(0)}{x} + \frac{\angle 1}{\angle 3}\frac{f^{(3)}(0)}{x^3} + \frac{\angle 2}{\angle 5}\cdot\frac{f^{(5)}(0)}{x^5} + \ldots.\right\} \tag{8.1}$$

Putting x for x^2 and y for y^2 we find that

$$\int_0^\infty \frac{f(\sqrt{y})}{\sqrt{y}}\,e^{xy}dy = \sqrt{\frac{\pi}{x}}\left\{f(0) + \frac{1}{\angle 2}\cdot\frac{1}{2}\frac{f^{(2)}(0)}{x} + \frac{1}{\angle 4}\cdot\frac{1}{2^2}\frac{f^{(4)}(0)}{x^2} + \ldots.\right\} + 4\left\{\frac{f^{(1)}(0)}{x} + \frac{\angle 1}{\angle 3}\frac{f^{(3)}(0)}{x^2} + \frac{\angle 2}{\angle 5}\frac{f^{(5)}(0)}{x^3} + \ldots.\right\}. \tag{8.2}$$

Hence from (3.2) and (8.2), it follows that the series

$$\sqrt{\frac{\pi}{x}}\left\{f(0)+\frac{1}{\angle 2}\cdot\frac{1}{2}\frac{f^{(2)}(0)}{x}+\frac{1}{\angle 4}\frac{1}{2^2}\frac{f^{(4)}(0)}{x^2}+\ldots.\right\}$$
$$+4\left\{\frac{f^{(1)}(0)}{x}+\frac{\angle 1}{\angle 3}\frac{f^{(3)}(0)}{x^2}+\frac{\angle 2}{\angle 5}\frac{f^{(5)}(0)}{x^3}\right.$$
$$\left.+\ldots.\right\}, \tag{8.3}$$

is R_c (R_s) if the function

$$\frac{f(\sqrt{y})}{\sqrt{y}}, \tag{8.4}$$

is R_s (R_c). In particular if

$$f(\sqrt{y})=1, \tag{8.5}$$

we find that (8.4) reduces to the well known function[2]

$$\frac{1}{\sqrt{y}} \tag{8.6}$$

which is R_s (R_c). Accordingly (8.3) also reduces to the function

$$\frac{1}{\sqrt{x}}, \tag{8.7}$$

which, then becomes R_c (R_s).

9. Finally it has also been shown by Dr. Willis[3] that the Asymptotic expansion of the function

$$\int_0^\infty e^{\frac{1}{2}y^2}J_0(xy)\,dy, \tag{9.1}$$

is

$$\frac{1}{x}+\frac{1^2}{\angle 1}\frac{1}{x^3}+\frac{1^2.3^2}{\angle 2}\cdot\frac{1}{x^5}+\frac{1^2.3^2.5^2}{\angle 3}\frac{1}{x^7}+\ldots. \tag{9.2}$$

Putting $\sqrt{2}x$ for x, and $y/\sqrt{2}$ for y, we find that the Asymptotic expansion of

$$\frac{1}{\sqrt{2}}\int_0^\infty e^{\frac{1}{2}y^2/2}J_0(xy)\,dy, \tag{9.3}$$

is given by

$$\frac{1}{2^{1/2}x}+\frac{1^2}{\angle 1}\cdot\frac{1}{2^{3/2}x^3}+\frac{1^2.3^2}{\angle 2}\cdot\frac{1}{2^{5/2}x^5}+\frac{1^2.3^2.5^2}{\angle 3}\frac{1}{2^{7/2}x^7}+\ldots. \tag{9.4}$$

Also Dr. Brij Mohan[2] has shown that

$$e^{x^2/2}, \tag{9.5}$$

is R_C. Hence from (9.5) and (5.2) it follows that

$$g(x) = \frac{1}{\sqrt{2}} \int_0^\infty e^{y^2/2} J_0(xy)\, dy, \tag{9.6}$$

is R_g.

Also, it has been shown by Willis[3] that

$$\int_0^\infty e^{y^2} J_0(xy)\, dy = \sqrt{\pi}\, e^{x^2/8} I_0\left(\frac{x^2}{8}\right),$$

so that

$$\int_0^\infty e^{y^2/2} J_0(xy)\, dy = \sqrt{2\pi}\, e^{x^2/4} I_0\left(\frac{x^2}{4}\right). \tag{9.7}$$

Hence from (9.7), (9.6) and (9.4), we conclude that the R_s function

$$e^{\frac{-x^2}{4}} I_0\left(\frac{x^2}{4}\right) \tag{9.8}$$

has its Asymptotic expansion given by the series

$$\sqrt{\frac{2}{\pi}}\left[\frac{1}{2x} + \frac{1^2}{\angle 1}\frac{1}{2^2x^3} + \frac{1^2.3^2}{\angle 2}\frac{1}{2^3x^5} + \frac{1^2.3^2.5^2}{\angle 3}\frac{1}{2^4x^7} + \ldots\right]. \tag{9.9}$$

I offer my thanks to Dr. Brij Mohan for his help and guidance in the preparation of this paper.

REFERENCES

1. Brij Mohan .. "A class of kernels," *J. of Banaras Hindu University*, Silver Jubilee Number, 1942, 134–37.

2. ——— .. "A brief history of Self-Reciprocal functions," *J. of Indian Math. Soc.*, 1935, **1** (7), 211.

3. Willis .. "A formula for expanding an integral as a series," *Phil. Mag. 7th Series*, Jan.–Dec. 1948, **39**, 455–60.

LĪLĀVATĪ—A NEW ANALOGUE COMPUTER FOR SOLVING LINEAR SIMULTANEOUS EQUATIONS AND RELATED PROBLEMS

Part II. Design of Model II and its Application to the Solution of Secular Equations

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1. Introduction

The general principles involved in the design of a new analogue computer (Līlāvatī) for solving linear simultaneous equations were discussed in a previous paper (G. N. Ramachandran and E. V. Krishnamurthy 1958—referred to as Part I). It was shown there how, unlike other computers where loading effects introduce errors due to current drain and non-linearity, this computer could provide an exact analogy of the set of equations to be solved, by making use of Ohm's law for multiplication and null methods for comparison of voltages. Even though these were the methods used in Model I for setting up the analogy, the measurements of voltages and currents were only done by meters, which are in principle not capable of highest precision. If exact measurements are to be made, only potentiometric methods should be used and meters should be completely dispensed with. Quite apart from the question of accuracy, the potentiometric arrangement has also other advantages;

(*a*) Considerable economy is possible in material and in number of operations;

(*b*) The circuit can easily be modified to solve secular equations;

(*c*) Operations can be made both with a.c. and d.c. without changing the circuit appreciably.

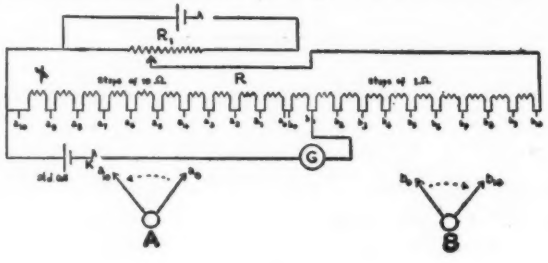
This paper primarily deals with the design of Model II of Līlāvatī, employing potentiometers and its applications. Still further improvements are possible if one makes use of different iterative methods for solving both

simultaneous and secular equations and if corresponding changes in circuitry are made. A discussion of these is reserved for Part III.

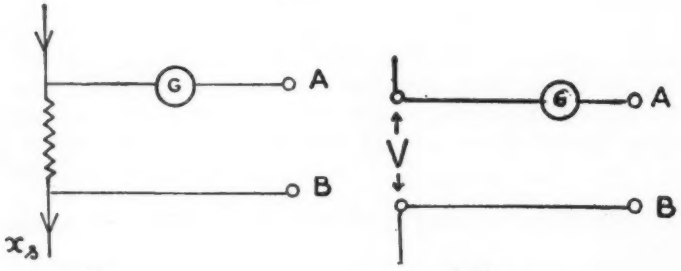
2. THE POTENTIOMETRIC ARRANGEMENT

The essential idea, as was mentioned in Part I, is to measure both currents and voltages by means of a standard potentiometer, the latter directly and the former by incorporating a standard resistance in the current circuit and measuring the voltage developed across it. In building Model II, the main aim was to test the suitability of potentiometric circuits in the design of the computer and consequently all components were designed only to have an accuracy of two significant figures (or 1%).

The potentiometric arrangement itself is shown in Fig. 1 (*a*) and it could be used for measuring any voltage between 0 and 1·1 Volts to an



1 (*a*)



1 (*b*) 1 (*c*)

FIGS. 1 (*a*), 1 (*b*) and 1 (*c*). Potentiometric arrangement for measuring current and voltage.

accuracy of 0·01 V. It consists of a two-decade resistance box (R), the current through which could be varied by means of a variable resistance R_1. In order to standardise it, a standard Weston cell is connected across 101 Ω, with a tap key K and a galvanometer G. If the key K is pressed and the rheostat R_1 is adjusted for no current in G, then the potentiometer is standardised.

By means of two 11-way switches A and B, whose 11 ways are connected respectively to the points a_0 to a_{10} and b_0 to b_{10} of the two-decade resistance box, any voltage value between 0 and 1·1 V. can be obtained across A and B, according to the setting of the switches A and B.

In order to measure the current, a standard resistance of 100 Ω (r) is included in the current circuit and the potential drop across it is compared with that between the points A and B of the potentiometer as shown in Fig. 1 (*b*). The switches A and B are adjusted for balance and their setting gives straightaway the currents in units of 0·1 m Amp. The same method can be used to measure any voltage in the range 0 to 1·1 V. in units of 0·01 V. [Fig. 1 (*c*)].

3. The New Circuit

In order to solve a set of equations:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}\,x_n &= b_1 \qquad (1)\\ a_{21}x_1 + \cdots + a_{2n}\,x_n &= b_2\\ \cdots\cdots\cdots\

B, B_1 .... B_n as shown in the figure. Thus the voltages b_1 to b_n can assume any value between $-1{\cdot}10$ V. to $+1{\cdot}10$ V. in steps of $0{\cdot}01$ V. depending on the setting of the corresponding switches, which can be readily read off on dials.

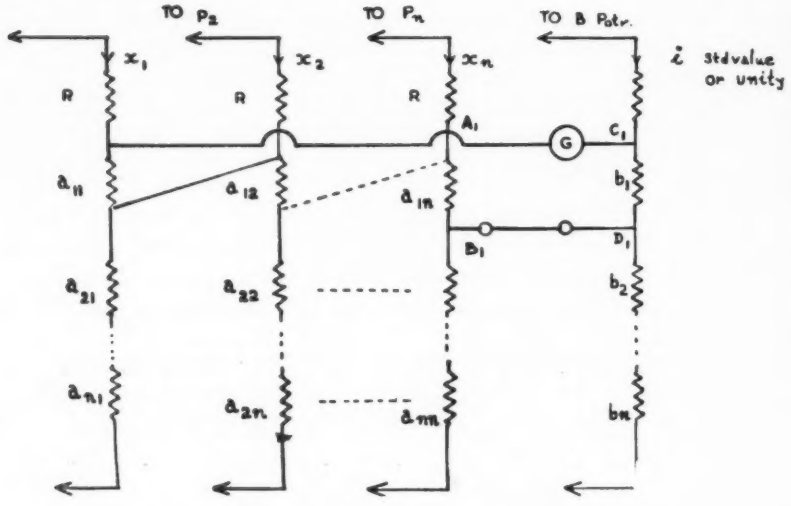


FIG 2. Analogy of a set of *n*-linear simultaneous equations.

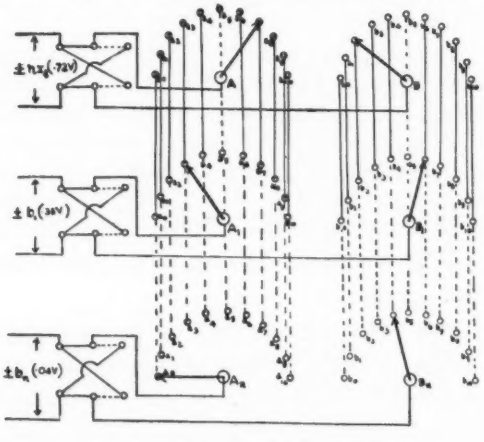


FIG. 3. The setting switches.

The cross-connections needed for setting up one equation at a time are shown in Fig. 2. The connections there are shown as having been made

for the first row. By means of a selector, as described in Part I [§4, (*iv*),] such connections can be made successively corresponding to each row of the equation from 1 to n. The currents x_1 to x_n are adjusted, as before, for balance at each step. The value of the currents need not be read at intermediate stages, but they can be accurately measured by means of the switches A and B, when the iterative process has converged.

Thus in Model II, a single potentiometer can not only be used to obtain the voltage settings corresponding to the values b_r on the right hand side, but also for measuring the currents x_s. Further, the technique of measurement is based on a null setting, so that it is capable of great accuracy in principle.

In §2, the method of standardising the potentiometer was described. In principle, there is no need to standardise the potentiometer, if the same potentiometer is used for setting the voltages b_r and for measuring the currents x_s. This is so since the value of the current is obtained from a measurement of the voltage drop over a known resistance, so that both the voltage values (b_r) and the current values (x_s) are affected in the same proportion if the current through the potentiometer is different from the standard value. The equations (1) are still true, since the resistance values a_{rs} remain unchanged. Of course, this argument is valid only if the current through the potentiometer remains steady during the process of solving the equations.

However, if a dry battery is used for feeding the potentiometer, the e.m.f. may gradually fall during operation, thereby altering the current through the potentiometer. In such a case, it is necessary to keep the voltage across the ends of the potentiometer steady, by periodically standardising it against some reference voltage.

If a.c. is used, the need for standardisation can be completely eliminated provided the sources of e.m.f. for all the circuits are derived from the same primary source by means of transformers (*see* §5, below).

4. Details of the Model II (Fig. 4)

(*i*) *Description of the model*

The components used for building Model II of the computer were almost similar to those of Model I. The 100 Ω coefficient volume controls (a_{rs}) are mounted on the left hand side of the top panel (to the left of the galvanometer) and the sign switches for these coefficients are located on the right hand side. The toggle switch, at the top left of the galvanometer, is the one that is used for removing the shunt. The switch to the right of

this is the off-on switch for the voltage potentiometer. At the left of the lower panel are the on-off switches for the three current circuits and the *plus-minus* switches for currents x_s. Below these are seen the P-potentiometers P_1, P_2 and P_3 (with white knobs), for varying these currents.

The multipole setting switches for b_r (A_1, B_1; A_2, B_2; A_3, B_3) are located on the right of the lower panel, along with their sign reversal switches.

The two knobs at the bottom are the setting switches A, B for measuring the currents x_s.

The selector switch S is at the top middle of the lower panel. The galvanometer is brought to the Wheatstone's network in the first position of this switch (marked W) and in the second position (marked S) the galvanometer is brought to the standardising circuit. In the next three positions of this selector (marked 1, 2 and 3) the iterative steps are carried out.

The two-decade resistance box is fixed to the left-side of the computer. The 2 pole 9 way selector for bringing each of the nine a_{rs} to the fourth arm of the Wheatstone's bridge and the variable resistance (R′) that controls the current through the voltage potentiometer are mounted on the right side of the computer (not seen in the photograph).

(*ii*) *Auxiliary circuits*

(*a*) *Circuit for setting resistances.*—The auxiliary circuits used in Model II were similar to those in Part I.

The same two-decade resistance box R which is used for the potentiometer forms the third arm of the Wheatstone's network. A D.P.D.T. switch located on the right side of the instrument enables one to isolate this box from the Wheatstone's network while it is being used as a potentiometer. When it is switched off, the two contacts K_1, K_2 in Fig. 5 (*b*) are broken. A 2 pole 9 way selector switch brings each one of the volume controls (a_{rs}) to the fourth arm of the Wheatstone's network. The galvanometer is brought to this network by turning selector S to the position W.

(*b*) *Standardisation of the potentiometer.*—In the last section, it was mentioned that while using a dry battery, one should keep the current through the potentiometer constant. For this purpose, a standard Weston cell is connected across 101 Ω of the box, through the galvanometer, which may be done by bringing the selector switch to the position marked S. The variable resistance R_1 at the right side is adjusted for null reading of the galvanometer. The standardisation can be checked whenever necessary by bringing the selector S to this second position (S).

(*c*) *The sign switches.*—The sign switches for affixing *plus* or *minus* signs to x_s, b_r and a_{rs} are exactly similar to those mentioned in Part I §4, (*i*).

(*d*) *The equation selector.*—The equation selector used is an 8 pole 8 way ganged switch. The function of this switch will well be understood from Fig. 5 (*a*). The terminals marked C, D, E, F, G, H, I and J are the 8 poles of the switch and c_i, d_i, e_i, f_i, g_i, h_i, i_i and j_i ($i = 1$ to 8) are the eight ways corresponding to each one of these poles.

For example, if this selector is set at the 4th position, since each pair of the poles (C, D), (E, F) and (I, J) are shorted, the points (c_4 and d_4), (e_4 and f_4) and (i_4 and j_4) will be connected respectively, and the galvanometer will be connected between the points g_4 and h_4.

It is easy to see from the figure that at each of the settings 3, 4 and 5 it connects up a particular row and also connects the galvanometer for balancing against the corresponding b_r. The iterative process is thus carried out in the 3rd, 4th and 5th positions of this switch (marked 1, 2, 3 in Fig. 4).

Fig. 5 (*b*) and 5 (*c*) show how the same galvanometer is brought to the Wheatstone's network and to the standardising circuit, in the first and second positions respectively of the selector switch.

The 6th, 7th and 8th positions of this selector are used for reading off the currents x_1, x_2 and x_3 [Fig. 5 (*d*)].

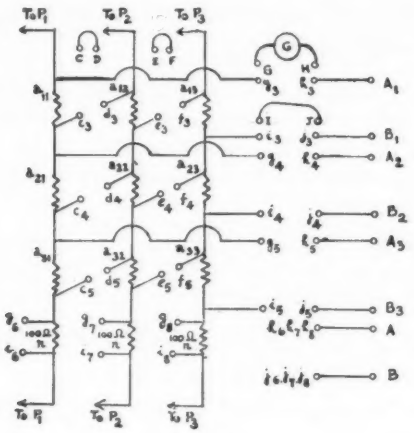
(*e*) *Current measurements.*—A standard 100 Ω resistor is included in each one of the circuits and the ends of each one of these resistors can be brought for comparison against points A and B of the voltage potentiometer (R) successively, in the 6th, 7th and 8th positions of the selector [Fig. 5 (*d*)]. The switches A and B could then be adjusted each time for balance in the galvanometer and the current readily read off on the dials A and B.

(*f*) *The galvanometer.*—A 0–500 μA ammeter with a resistance of 100 Ω is used as the galvanometer with a shunt of 100 Ω, which could be removed when needed by means of a switch. The sensitivity of this galvanometer was found to be sufficient to get an accuracy within 1%.

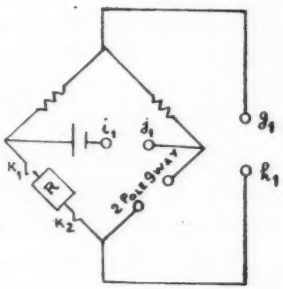
5. Operation with A.C.

As mentioned earlier, the arrangement described in this paper can be used with an alternating voltage supply which would replace all the batteries. Assuming that there is no inductive effect, there will not be any phase change introduced due to the current flow in the various circuits. If the source

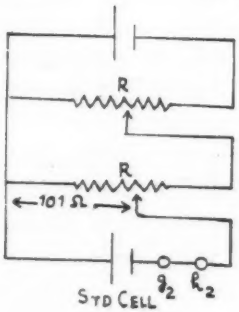
of alternating voltage for the various circuits, including the potentiometric circuit, are derived from a number of secondaries of the same transformer, then the currents would all be either in phase or exactly out of phase according to the way in which they are connected to the secondary terminals. Therefore, if a commutator is incorporated in this circuit and it is reversed, the effect with a.c. is exactly equivalent to reversing the sign of the current with d.c. Consequently, all the arrangements used in the d.c. model for changing the sign of x_s, b_{rs} and a_{rs} can be adopted, without any further modification, for the a.c. model also.



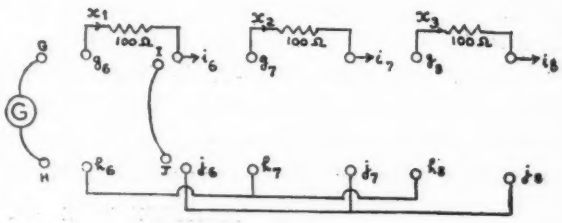
5 (*a*)



5 (*b*)



5 (*c*)



5 (*d*)

FIGS. 5 (*a*), 5 (*b*), 5 (*c*) and 5 (*d*). Circuit diagram of the selector switch.

The only component that needs a change is the null indicator which should be an a.c. galvanometer.

In testing out Model II, with a.c., a 0–100 μA meter with a germanium crystal rectifier was used as a null indicator and was found to serve the purpose very well. A step-down transformer with a number of secondary coils (220 V.–6·3 V.) was used as the source of current for the various circuits. Consequently the fluctuations in the mains voltage equally affect all the ciruits, so that no errors are introduced. There is no need therefore, for standardisation. In practice, it was found that, even when heavy fluctuations of the order of 100 Volts in the input voltage (150–250 V.) were introduced artificially by using a Variac, the computer worked quite satisfactorily, *viz.*, the balance point was not upset by such fluctuations.

Even though the wire wound potentiometers used were not strictly non-inductive, the error introduced was less than 1%. If one desires to have higher accuracy, it would be necessary to use non-inductive resistances, *i.e.*, a good resistance box, for each coefficient a_{rs}. This would require n^2 resistance boxes, which would make the unit very costly (although it eliminates the need for a Wheatstone's network). However, since only one row is connected up at a time, it is possible to devise an arrangement with only n resistance boxes, by having a series of n^2 setting switches, similar to those used for setting the n voltages b_r, from a single potentiometer. Such an arrangement is proposed to be built up and will be described in Part III. For this purpose an arrangement of resistances similar to the Kelvin-Varley potentiometer is required, rather than the ordinary two or three-decade resistance boxes.

6. Modification to Solve Secular Equations

Of frequent occurrence in many fields is the problem of determining those values of a constant λ for which non-trivial solutions exist to the homogeneous set of equations

$$\begin{aligned} a_{11}x_1 + \cdots\cdots\cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + \cdots\cdots\cdots + a_{2n}x_n &= \lambda x_2 \\ \cdots\cdots\cdots\cdots\cdots & \\ a_{n1}x_1 + \cdots\cdots\cdots + a_{nn}x_n &= \lambda x_n \end{aligned} \tag{2}$$

or, in matrix notation

$$\mathrm{AX} = \lambda \mathrm{X} \tag{3}$$

Such a problem is known as a characteristic value problem; values of λ for which nontrivial solutions exist are called characteristic values (also eigenvalues or latent roots) of the problem. These values of λ are also referred

to as the eigenvalues of the matrix A, *i.e.*, the matrix composed of the coefficients a_{rs}. The corresponding vector solutions X (having components x_s) are known as the characteristic vectors (or eigenvectors) of the problem or of the matrix A.

Equation (3) can also be written in the form

$$(A - \lambda I)\, X = 0 \tag{3a}$$

where I is the unit matrix of order n. This homogeneous problem possesses nontrivial solutions if and only if the determinant of the coefficient matrix, *i.e.*, $|A - \lambda I|$, vanishes or

$$|A - \lambda I| = 0 \tag{4}$$

Expanding this determinant, one finds that λ is a root of an algebraic equation of degree n, known as the secular equation. The n solutions $\lambda_1, \lambda_2, \ldots . \lambda_n$ which need not all be distinct, are the characteristic values or latent roots of the matrix A.

It is to be noted that if any one of these values of λ are substituted in (2) and solved for $x_1 \ldots\ldots x_n$ only $(n-1)$ equations will be independent so that only $(n-1)$ ratios are relevant to the problem.

A more general eigenvalue problem is to find the values of λ which satisfy the matrix equation

$$AX = \lambda BX. \tag{5}$$

However this will not be discussed here.

Model II can easily be modified to obtain the eigenvalues of equation (2) by using an iterative method described by Frost and Tamres (1947). But it cannot readily be adapted to solve the more general type of equations (5). The various methods of solving equation (5) and their relative merits as far as they can be mechanised are reserved for Part III. Here the aim is to show that a simple modification of the circuit discussed above can be used to solve equations of the type (2).

We know that the left-hand side of equation (2) can be set up in analogy. If now this equation is rewritten in a slightly different form, *viz.*,

$$\begin{aligned}
&(a_{11} - \lambda)\, x_1 + \quad \cdots\cdots\cdots + a_{1n}x_n = 0 \\
&a_{21}x_1 + \quad \cdots\cdots\cdots + a_{2n}x_n = 0 \\
&\quad \cdots\cdots\cdots \\
&a_{n1}x_1 + \quad \cdots\cdots\cdots + (a_{nn} - \lambda)\, x_n = 0
\end{aligned} \tag{6}$$

it will be seen how a circuit analogy of the complete equation can be set up. It is only necessary to have a method whereby the value of the resistances representing the diagonal terms $a_{11}, \ldots\ldots a_{nn}$ can all be decreased (algebraically) by the same magnitude.

This could conceivably be arranged by using ganged potentiometers for these resistances. In that case, one method of solving (3) will be to start with arbitrary values $x_1 = 1$, x_2, $x_3 \ldots\ldots x_n$ for the various currents and $\lambda^{(0)}$ of λ and then to make use of the following iterative process (Frost and Tamres, 1947).

(i) Set up the first equation and change $x_2^{(0)}$ to $x_2^{(1)}$ for balance.

(ii) Set up the second equation and change $x_3^{(0)}$ to $x_3^{(1)}$ for balance.

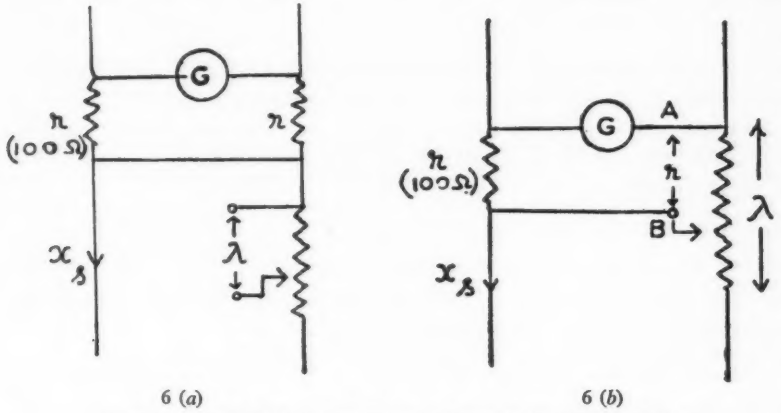
(iii) Continue until the $(n-1)$-th equation is set up and adjust the n-th variable $x_n^{(0)}$ to $x_n^{(1)}$ for balance.

(iv) Set up the n-th equation and adjust $\lambda^{(0)}$ to $\lambda^{(1)}$ for balance.

(v) Repeat the whole sequence until the quantities x_2, $x_3 \ldots\ldots x_n$, λ remain constant within the desired limits.

However it is difficult to gang potentiometers accurately and so a different method was adopted for Model II. In this method, an exact analogy of equation (2) is set up. Here the left-hand side is exactly the same as for the solution of simultaneous equations. On the right-hand side, however, we do not have the potentiometers giving voltages $b_1 \ldots\ldots b_n$ but a single potentiometer (λ) whose resistance can be set equal to λ and the current through which can be made equal to any required value from x_1 to x_n. The latter adjustment can readily be made by incorporating a standard resistance r (100 Ω) in series with the λ potentiometer and comparing the voltage developed across this with an equal resistance r (100 Ω) kept in series with each of the current circuits x_1 to x_n (Fig. 6 *a*) [the latter are already there in Model II, see §4, ii (*e*)]. However, it is not necessary to have a series resistance along with the λ-potentiometer; for one can tap off the required resistance r (100 Ω) directly from the λ-potentiometer as shown in Fig. 6 (*b*), and bring it for comparison, so that equalisation of the current can be made. The advantage of having this arrangement is that the setting switch A–B (which is used for measuring the currents if the potentiometer is standardised) can directly be made to serve this purpose, provided it is adjusted to read the value r (100 Ω). In order to set up the analogy of the right-hand side of all the equations (2), we need only a single potentiometer, through which the current flowing can be made different each time. So instead of having all the n ($A_n - B_n$) setting switches any one setting switch (say $A_1 - B_1$)

may be chosen for the purpose, the current through which being made equal to any one required value by the above method. Also, the value of λ can



6 (*a*) 6 (*b*)

FIGS. 6 (*a*) and 6 (*b*). Equalisation of current in λ-potentiometer.

be directly read off on the dials. These arrangements are easily made, if a slight modification is made in the connections to the selector switch S, shown in Fig. 5 (*a*). The modified circuit is shown in Fig. 7.

The terminals C, D, E, F, G, H, I and J are the 8 poles of the switch and c_i, d_i, e_i, f_i, g_i, h_i, i_i and j_i ($i = 1$ to 8) are the 8 ways. We will now see how the currents are equalised using this selector. Since C and D, E and F, and I and J are shorted and the galvanometer is connected between G and H, if the switch is set to the third position the points (c_3 and d_3), (e_3 and f_3), (i_3 and j_3) are respectively connected and the galvanometer is introduced between g_3 and h_3. This serves to make the current through the λ-potentiometer equal to x_1, the current through the first circuit. Similarly, at settings 5 and 7, one can make the current through the λ-potentiometer equal to x_2 and x_3. At these same positions 3, 5 and 7, it is also possible to measure the currents x_1, x_2, x_3 if the potentiometer is standardised. Thus the following will be the sequence of operations that will have to be made in order to solve equations (2) by the iterative process mentioned above.

(*a*) *Selector is set in position* 1.—The coefficients a_{rs} are all fed in as resistances using the Post Office Box arrangement and the signs of a_{rs} are also set.

(*b*) *Selector is set in position* 2.—The potentiometer is standardised by adjusting R_1 [as in §4, (ii)].

(*c*) *Selector is set in position* 3.—The current through the first circuit is measured by means of the setting switch A–B and adjusted to some standard value or unity (say x_1).

(*d*) The setting switch A–B is set to read 100 Ω and with selector in position 3, the variable resistance R_1 is adjusted for balance in the galvanometer. Thus the current x_1 is made to flow through the λ potentiometer.

(*e*) The other variables x_2 and x_3 are represented as arbitrary currents $x_2^{(0)}$ and $x_3^{(0)}$ need not be measured and in any arbitrary value for λ, say $\lambda^{(0)}$ is set on the switch A_1–B_1.

(*f*) *Selector is set in position* 4.—Potentiometer P_2 is adjusted and $x_2^{(0)}$ is varied to $x_2^{(1)}$ for balance.

(*g*) *Selector is set in position* 5.—The variable resistance R_1 is adjusted for balance making $x_2^{(1)}$ to flow in the λ-potentiometer.

(*h*) *Selector is set in position* 6.—Potentiometer P_3 is adjusted and $x_3^{(0)}$ varied to $x_3^{(1)}$ for balance.

(*i*) *Selector is set in position* 7.—The variable resistance R_1 is adjusted for balance making $x_3^{(1)}$ to flow in the λ-potentiometer.

(*j*) *Selector is set in position* 8.—The setting switch A_1–B_1 is adjusted for balance to get a better value for $\lambda^{(0)}$, namely $\lambda^{(1)}$.

(*k*) The selector is now brought back to position 3 and R_1 is adjusted for balance in galvanometer, thus starting a new cycle of operations.

These cycles of operations are continued until the quantities x_2, x_3 and λ remain constant within the desired limits. Then the values of x_2, x_3 are measured and λ can be read off from the setting of the switch A_1–B_1.

This iterative method presents no difficulties in converging to the largest and smallest eigenvalues. Once these eigenvalues are obtained, it is possible in principle, to force the convergence upon some other value by keeping the trial vector x orthogonal to both the largest and the smallest eigenvectors. But in practice, some difficulties in convergence to the intermediate eigenvalues may be encountered unless a good approximation is available to start with. This difficulty is felt even in numerical methods involving this particular iterative process. The method very largely depends on the experience of the computer and so automatic methods cannot be adopted (Murray, 1948). Fuller details on other iterative methods will be discussed in Part III.

The computer (Model II) can be present for solving simultaneous equations or for solving the secular equations by making use of a 6 pole 2 way

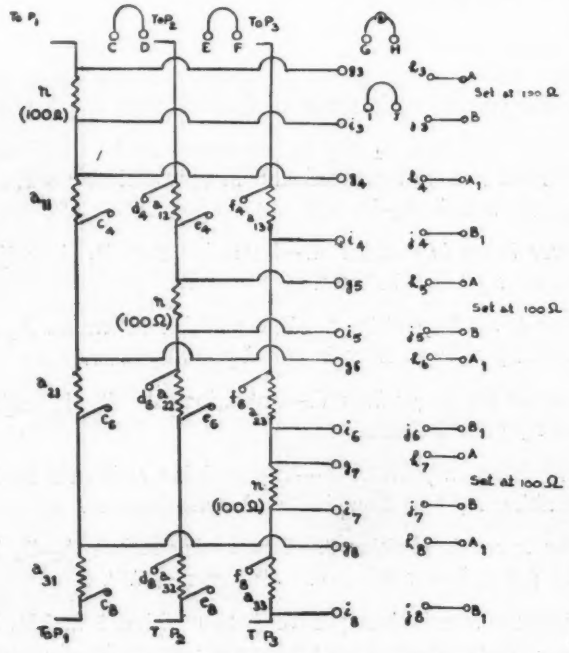


FIG. 7. Modified circuit for secular equations.

switch T (Fig. 8) (not seen in the photograph). When this switch is in its first way, it connects the points (h_4 and A_1), (j_4 and B_1), (h_6 and A_2), (j_6 and B_2), (h_8 and A_3), (j_8 and B_3) so that the computer can be used to solve simultaneous equations. When the switch T is in the second way, the computer

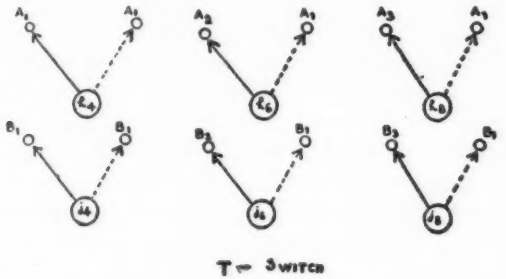


FIG. 8. The T-switch—sequence selector.

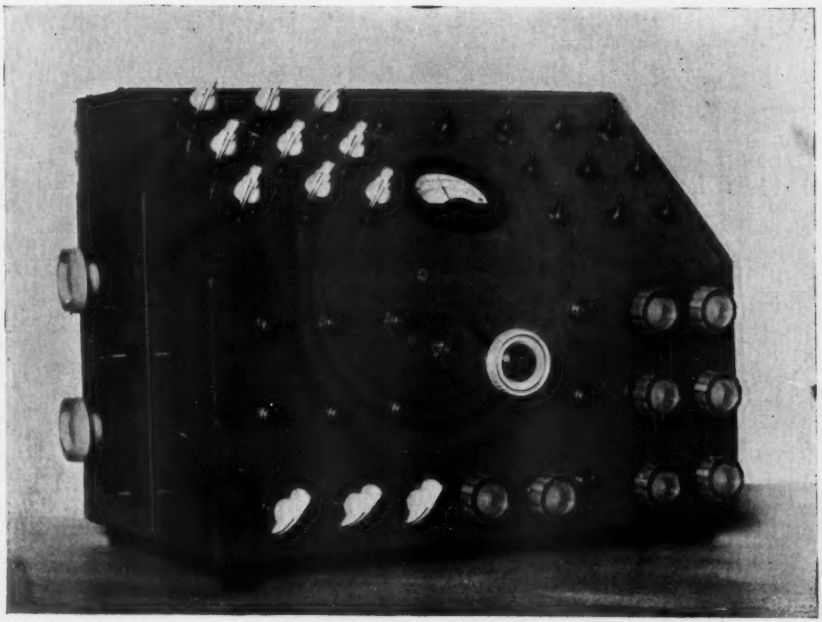


FIG. 4. Līlāvatī—Model II

is ready for solving secular equation, since all the points h_4, h_6 and h_8 are connected to A_1, and j_4, j_6 and j_8 are connected to B_1, as shown in Fig. 8.

As a result of this modification, namely of the computer being able to solve either simultaneous equations or secular equations, the settings of the selector switch at which the iterative process for solving the former are made, are slightly different from those described in §4, ii (*d*). Instead of iterating in the settings 3, 4, 5 and measuring the currents x_s at 6, 7 and 8, iteration is now performed at settings 4, 6 and 8 and currents are measured at 3, 5 and 7.

The computer described here can handle only a third order secular equation. However there is no essential difficulty in constructing a larger computer based on these principles by making use of switches having larger capacity.

7. Summary

This paper essentially deals with Model II of Līlāvatī (an analogue computer for solving linear simultaneous equations) where unlike in the first model, potentiometric methods are employed for measurements of current and voltage. The incorporation of the potentiometer has considerably simplified the number of operations required and has also helped in successfully modifying this computer to solve characteristic value problems. The operational details of this computer are also described in this paper.

8. Acknowledgement

The author wishes to express his deepest sense of gratitude to Prof. G. N. Ramachandran for the helpful guidance and encouragement received during the course of these investigations. Thanks are also due to Mr. V. R. Sambandam, Mechanic, for the help rendered in the construction of this instrument.

9. References

Frost, A. A. and Tamres, M. .. *J. Chem. Phys.*, 1947, **15**, 383.

Murray, P. F. J. .. *Quart. Appl. Math.*, 1948, **7**, 263.

Ramachandran, G. N. and Krishnamurthy, E. V. *Proc. Ind. Acad. Sci.*, 1958, **48**, 152.

SOME POLISHING EXPERIMENTS ON COPPER IN A HYPERBOLIC CELL

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Abstract

Some polishing experiments have been carried out on copper anodes in a hyperbolic cell designed by Gilmont and Walton, using orthophosphoric acid as the electrolyte. The results obtained have been compared to those obtained in similar experiments in a Hull cell. It has been found that very similar bands of different reflectivity and polishes are found to form in both the cells. These bands shift with time and a study of such displacements has been made. The results are briefly discussed.

Introduction

Of late various workers[1–5] have tried to design and work out the theory of different types of electrolytic cells which would give a uniformly varying current density along an electrode assuming that there is no polarization, a condition normally not attainable in practice. The use of the Hull[1,2] cell for studying electro-plating phenomena is well known. The construction of the Hull cell is fairly simple, but a theoretical and mathematical analysis of the current distribution is not so easy. Although recently Skwirzynski and Huttly[3] have given a mathematical solution of the current distribution for a trapezoidal cell very much similar to Hull cell, yet they found that the current distribution is linear only on about $\frac{2}{3}$ of the electrode, there being a singularity at the high current density end. According to Kasper,[4] the proper choice of electrode geometry should give a sufficiently linear current distribution with simple mathematical solutions which would eliminate the need for emperical methods. Recently, Gilmont and Walton,[5] on the basis of Kasper's ideas, have designed a new cell, usually referred to as hyperbolic cell, in which the current density function along an electrode is almost linear and can be predicted by a simple equation. The correct geometrical shape of the new cell was obtained by applying the fundamental principles of the theory of electro-potentials, according to which the current density lines are orthogonal to the equi-potential surfaces.

Briefly stating, the hyperbolic cell (Fig. 1) as designed by Gilmont and Walton consists of 4 sides, two of which are straight and equal intersecting at 45°. The other two sides are rectangular hyperbolas orthogonal to each other and each in turn orthogonal to one of the straight sides. One of the straight sides and its opposite curved surface are used as electrodes. The straight side is used as the anode while the curved side as the cathode. Recently, we have reported[6,7] some of our experimental findings on the electrolytic polishing of copper in a Hull cell using orthophosphoric acid as electrolyte. There we observed highly different finishes on one and the same specimen, with sharp contrasts and discontinuities between these finishes or bands. Also, it was found that two almost equally polished bands were always separated by a small unpolished band. The position of the various bands was also found to shift with the time of polishing. As no definite and clear insight into these phenomena is so far available and as the current distribution in the Hull cell is rather complex, it was considered worth while to compare these observations with similar experiments using the hyperbolic cell. The results of these comparative experiments in a hyperbolic cell are reported below.

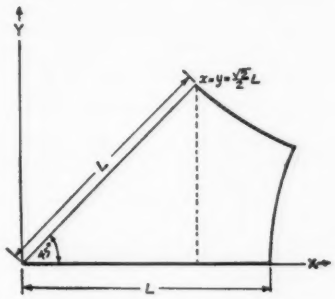

FIG. 1

EXPERIMENTAL

In order to facilitate comparison, the hyperbolic cell was also made of perspex and the electrolyte used was of the same concentration (756 g/l of H_3PO_4) as employed in our Hull cell studies. The experimental details and copper (high purity 99·98%) electrodes were similar to those used in our previous study.

RESULTS AND DISCUSSION

Figs. 2, 3, 4 and 5 show the various effects produced on copper electropolished in the hyperbolic cell for varying times. It will be seen that in

all the figures there are 5 well-defined regions showing different reflectivity, colour, structure etc. The same are briefly described and shown in Fig. 6. The above results were found to be quite reproducible both as regards their nature and their relative position. The bands are also found to be slightly curved at the top towards the high current density end and straight at the bottom. The most interesting observation was that although this type of cell is known theoretically to have a very linear current distribution, yet, the two well polished regions (II and IV, Fig. 6) were always found to be separated by a small unpolished or frittered region (III), exactly as was the case in Hull cell experiments.

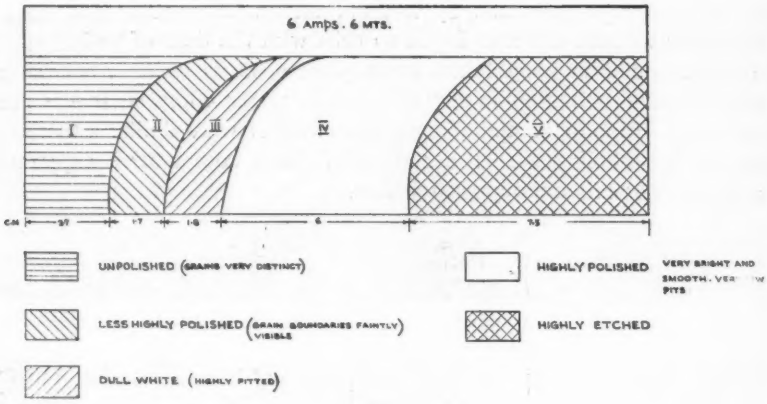

FIG. 6

The shift in the position of bands with varying time also similarly compares with that observed with Hull cell, *i.e.*, as the polishing time increases, the different bands are found to shift towards the low current density end upto a certain extent of time after which they tend to retrace and shift back towards the high current density end. For example, during polishing at a total cell current of 6 A, the bands shift towards the low current density end upto a period of about 6 min. If the polishing time is further increased beyond 6 min. the shifts are in the opposite direction (*see* Figs. 4 and 5) as the time of polishing increases. Similarly, for Hull cell, it was found that at a total cell current of 4 A, the retracing of the band-shifts occurs after about 8 min.

Thus, we see that on the whole the results obtained both in a hyperbolic cell and in a Hull cell resemble each other quite closely. The full logic of these observations is not yet clear. It appears that for understanding the

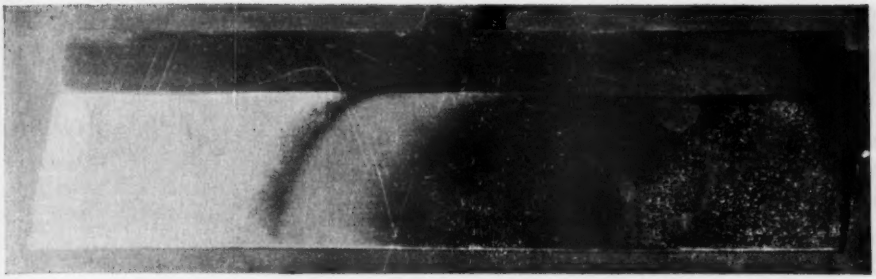

FIG. 2. 6 Amps. 3 Mts.

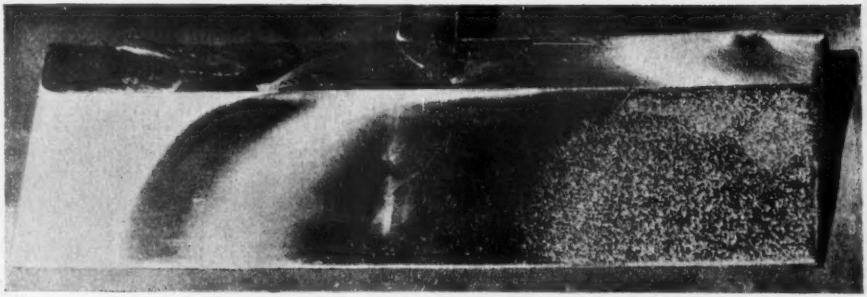

FIG. 3. 6 Amps. 6 Mts.

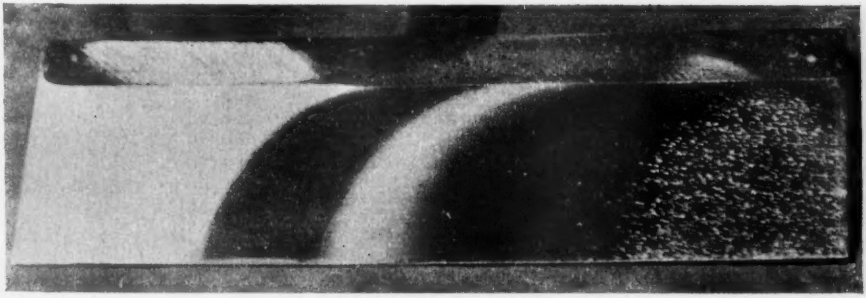

FIG. 4. 6 Amps. 12 Mts.

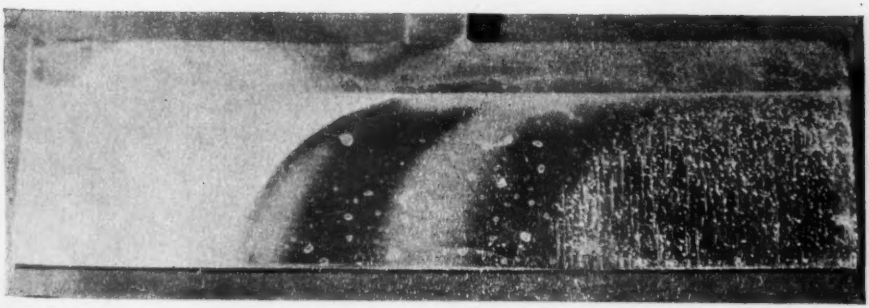

FIG. 5. 6 Amps. 24 Mts.

above phenomena, we must look for a detailed knowledge of (*a*) the progressive effect of the anodic reaction on the composition of the film of liquid near the anode, (*b*) the formation of solid films on the anode and their electrical properties, (*c*) the effect of the resulting intense polarization on the current density pattern on the anode as a function of time, (*d*) the current density values necessary for getting polishing effect under the particular hydrodynamic conditions obtaining at the anode, and (*e*) short-circuiting effect by the anode on the regions of different potentials and the consequent effect of current density pattern.

It appears that the actual conditions existing in such cells are very different from those initially assumed for working out the cell design. In electropolishing, in particular, there is a high degree of polarization due to the production of poorly conducting anodic films. This leads to complete deviation of the current distribution on the electrode, from the theoretical function. Since, moreover, the polarization is a function of time and depends on the hydrodynamic conditions, the whole pattern of current density distribution would considerably change with time.

No detailed study of the above factors has so far been reported and further work is required for a correct elucidation of our observations.

Acknowledgment

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References

1. Hull, R. O. .. *Proc. Amer. Electroplaters' Soc.*, 27th Convention, 1939.
2. Cuthbertson, J. W. .. *Metal Industry*, August 1951, p. 87.
3. Skwirzynski, J. K. and Huttly, M. *J. Electrochem. Soc.*, 1957, **104**, 650.
4. Kasper, C. .. *Trans. Am. Electrochem. Soc.*, 1940, **77**, 353.
5. Gilmont, R. and Walton, R. F. *J. Electrochem. Soc.*, 1956, **103**, 549.
6. Sanghi and Ramanathan .. "Electropolishing of copper in orthophosphoric acid bath using a Hull cell"—presented at the Symposium of the India Section of Electrochemical Society, 1958.
7. ———— .. "Polarization and Polishing studies on copper in a Hull cell" —(under publication).

THE DIFFRACTION OF LIGHT BY SUPERPOSED PARALLEL SUPERSONIC WAVES GENERAL THEORY

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1. Introduction

The first theoretical investigations of the diffraction of light by superposed supersonic waves in liquids and solids date from 1938. Theories were then developed by Bergmann and Fues,[1] Nagendra Nath[2] and Nagabhushana Rao[3] and were mainly based on Raman and Nath's preliminary theory.[4] The considered sound waves have frequencies which are incommensurate and they do not present phase differences, so that the expressions for the intensities of the spectra derived by those authors only have a limited use.

Experiments showing combination spectra were performed by Bergmann[5] (in liquids), Hiedemann and Hoesch[6] (in solids), Bergmann and Fues[1] (in liquids and solids) and Govinda Rao[7] (in liquids). None of those experimental studies gives quantitative measurements of the intensities of the obtained spectra.

In 1949 Ramachandra Rao[8] set up a theory of the diffraction of light by two superposed parallel ultrasonic waves, the wavelength of the second being half the wavelength of the first (or the ratio of the frequency of the first to the frequency of the second being 1 : 2; there also exists a definite phase relationship between the two sound waves. Using the method of Raman-Nath's simplified theory and confining himself to a perturbation showing no time dependence, he comes to the interesting conclusion that the obtained spectra must be asymmetric with respect to the central image. This feature has been confirmed experimentally by the same author.

Those calculations were extended by Murty[9] to the case of two sound waves, the frequencies of which are generally in the ratio $1 : n$. This theory which is still based on Raman–Nath's elementary theory leads to the following conclusions:

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When the ratio of the frequencies is $1:(2l+1)$ the intensities of opposite orders are equal so that the pattern is symmetric;

When the ratio of the frequencies is $1:2l$ the intensities of opposite orders are different and the pattern is asymmetric.

Recently the interest in the problem of the diffraction of light by superposed parallel supersonic waves was again revived by the study of the distortion of finite amplitude sound waves by Zankel and Hiedemann[10]: the distorted wave is resolved into its Fourier components, so that the latter can be considered as superposed waves.

The purpose of the present paper is to expose the general theory of the diffraction of light by two superposed parallel supersonic waves, one being the n-th harmonic of the other. In the second section a system of difference–differential equations for the amplitudes of the different spectra is derived. In §3 the special case $\rho \equiv \lambda^2/\mu_0\mu_1\lambda^{*2} = 0$ is considered and the system of difference–differential equations is solved exactly. In the following section it is shown that in the case $\rho \neq 0$ the diffraction pattern is always asymmetric with respect to the zero order, as well for even as for odd values of n. Finally (§§ 5–7) a series solution of the general system is given for $n = 2$ and $n = 3$. In the latter case coefficients are calculated up to the term giving the asymmetry effect in the intensity.

It is a great pleasure for us to pay tribute to Prof. Sir C. V. Raman for the fundamental theories for the diffraction of light by supersonic waves he has established and which are characterized by the simplicity of the basic ideas underlying them. Both his preliminary and generalized theories were the first to give a satisfactory explanation of the diffraction patterns containing a great number of spectra and formed the basis for many theoretical and experimental studies, the present investigation included.

2. Establishment of a System of Difference–Differential Equations for the Amplitudes of the Diffracted Light Waves*

Let us consider a parallel beam of light of frequency ν and wavelength in vacuum λ, passing through a liquid column, disturbed by two superposed parallel progressive supersonic waves, one of them being the n-th harmonic of the other. We further suppose that the directions of propagation of the

* Independently and using the same method, Zankel[11] has obtained a system of difference-differential equations in the case where a complete set of harmonics, without phase differences, are contained in the sound wave. For this system, which is a generalization of our equation (9), he has found the exact solution for $\rho = 0$. If only one harmonic, containing a phase constant, is present Zankel has also obtained independently the exact solution (17) of Section 3, in the case $\rho = 0$.

light and sound waves are perpendicular to each other. We shall put the z-axis along the direction of the incident light and the x-axis along the direction of the sound waves.

In order to obtain the differential system for the amplitudes of the different spectra we shall follow the method of Raman and Nath's generalized theory.[12]

Writing for the electric field of the light

$$E(x, z, t) = e^{2\pi i\nu t}\phi(x, z, t), \tag{1}$$

$\phi(x, z, t)$ satisfies the equation

$$\Delta\phi = -\left(\frac{2\pi}{\lambda}\right)^2 \mu^2(x, t)\,\phi; \tag{2}$$

$\mu(x, t)$ is the refractive index of the disturbed liquid and is given by the expression

$$\mu(x, t) = \mu_0 + \mu_1 \sin 2\pi\left(\nu^* t - \frac{x}{\lambda^*}\right) + \mu_n \sin 2\pi\left(n\nu^* t - \frac{nx}{\lambda^*} + \Delta\right) \tag{3}$$

where

λ^* = wavelength of the supersonic waves in the considered medium;

ν^* = frequency of the supersonic waves;

μ_0 = refractive index of the undisturbed medium;

μ_1 = maximum variation of the refractive index of the disturbed medium due to the fundamental tone of the supersonic wave;

μ_n = maximum variation of the refractive index of the disturbed medium caused by the n-th harmonic of the supersonic wave;

Δ = phase of the second wave with respect to the first.

According to the periodicity of $\phi(x, z, t)$ in x and t, this function may be developed in a double Fourier series

$$\phi(x, z, t) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} f_{rs}(z)\, e^{2\pi i r\, x/\lambda^*} e^{2\pi i s\nu^* t} \tag{4}$$

Taking into account that $\phi(x + p\lambda^*, z, t) = \phi(x, z, t - p/\nu^*)$, p being any number, it follows that $f_{rs}(z) \equiv 0$ when $r \neq -s$ and $f_{rs}(z) \neq 0$ for $r = -s$.

Hence,

$$\phi(x, z, t) = \sum_{r=-\infty}^{\infty} f_r(z)\, e^{2\pi i r\, x/\lambda^*} e^{-2\pi i r \nu^* t}. \tag{5}$$

Substituting this series into the differential equation (2), equating the coefficients of equal exponentials on both sides, putting:

$$f_r = e^{-2\pi i\, \mu_0 z/\lambda} \phi_r(z) \tag{6}$$

into the result, and neglecting the second derivative of $\phi_r(z)$ with respect to the other terms, we finally obtain the following system of difference-differential equations,

$$2\frac{d\phi_r}{dz} - \sigma_1(\phi_{r-1} - \phi_{r+1}) - \sigma_n(\phi_{r-n}e^{-i\delta} - \phi_{r+n}e^{i\delta}) = i\beta r^2 \phi_r \tag{7}$$

with

$$\sigma_1 = \frac{2\pi\mu_1}{\lambda}, \quad \sigma_n = \frac{2\pi\mu_n}{\lambda}, \quad \beta = \frac{2\pi\lambda}{\mu_0 \lambda^{*2}}, \quad \delta = 2\pi\Delta \tag{7 a}$$

or putting

$$\xi = \sigma_1 z = \frac{2\pi\mu_1 z}{\lambda}, \quad \rho = \frac{\beta}{\sigma_1} = \frac{\lambda^2}{\mu_0 \mu_1 \lambda^{*2}}, \quad a_n = \frac{\sigma_n}{\sigma_1} = \frac{\mu_n}{\mu_1} \tag{8}$$

$$\boxed{2\frac{d\phi_r}{d\xi} - (\phi_{r-1} - \phi_{r+1}) - a_n(\phi_{r-n}e^{-i\delta} - \phi_{r+n}e^{i\delta}) = i\rho r^2 \phi_r.} \tag{9}$$

From (5) and (6) it is seen that the r-th order spectrum makes an angle θ_r, defined by $\sin\theta_r = -r(\lambda/\lambda^*)$, with respect to the z-axis, and undergoes a frequency change $-r\nu^*$; its amplitude is proportional with ϕ_r, the factor of proportionality having modulus one.

The boundary conditions are

$$\phi_r(0) = \delta_{r0} \tag{10}$$

where δ_{r0} is the well-known Kronecker symbol.

3. Special Case: $\rho = 0$

For great values of μ^1 and λ^* the right-hand side of equation (9) is negligible and the system becomes

$$2\frac{d\phi_r}{d\xi} - (\phi_{r-1} - \phi_{r+1}) - a_n(\phi_{r-n}e^{-i\delta} - \phi_{r+n}e^{i\delta}) = 0. \tag{11}$$

In order to integrate the system (11) with the boundary conditions (10) we shall use the complex function method of Nagendra Nath-Wilson.[3] We shall put

$$\phi_r(\xi) = \frac{1}{2\pi i} \oint F(\eta) \frac{e^{\xi f(\eta)}}{\eta^{r+1}} d\eta \tag{12}$$

where $F(\eta)$ and $f(\eta)$ are yet unknown functions of the complex variable η, and the integral is taken along a closed path in the complex η-plane, encircling the origin once in the positive sense. According to the boundary conditions (10) it is required that the function $F(\eta)$ must be

$$F(\eta) = 1. \tag{13}$$

Substitution of the expression (12) into the equation (11) gives

$$f(\eta) = \frac{1}{2}\left(\eta - \frac{1}{\eta}\right) + \frac{a_n}{2}\left(\eta^n e^{-i\delta} - \frac{1}{\eta^n e^{-i\delta}}\right) \tag{14}$$

so that the solution becomes

$$\phi_r(\xi) = \frac{1}{2\pi i} \oint \frac{\exp.\left[\frac{\xi}{2}\left(\eta - \frac{1}{\eta}\right)\right] \exp.\left[\frac{a_n \xi}{2}\left(\eta^n e^{-i\delta} - \frac{1}{\eta^n e^{-i\delta}}\right)\right]}{\eta^{r+1}} \times d\eta. \tag{15}$$

Using the well-known Laurent series for the exponential functions[13]

$$\exp.\left[\frac{\xi}{2}\left(\eta - \frac{1}{\eta}\right)\right] = \sum_{p=-\infty}^{\infty} J_p(\xi)\, \eta^p \tag{16 a}$$

$$\exp.\left[\frac{a_n \xi}{2}\left(\eta^n e^{-i\delta} - \frac{1}{\eta^n e^{-i\delta}}\right)\right] = \sum_{q=-\infty}^{\infty} J_q(a_n \xi)\, \eta^{nq} e^{-iq\delta}. \tag{16 b}$$

[$J_n(z)$ being the Bessel function of order n]
in the right-hand side of (15) we obtain

$$\phi_r(\xi) = \frac{1}{2\pi i} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} J_p(\xi)\, J_q(a_n \xi)\, e^{-iq\delta} \oint \eta^{p+rq-r-1}\, d\eta.$$

The integral being equal to $2\pi i \delta_{p,\, r-nq}$, we finally have

$$\boxed{\phi_r(\xi) = \sum_{q=-\infty}^{\infty} J_{r-nq}(\xi)\, J_q(a_n \xi)\, e^{-iq\delta}} \tag{17}$$

which are in fact the solutions obtained by Murty, using the method of Raman and Nath's preliminary theory, but written in a more condensed form. Calculating the intensities, Murty finds that in the case of odd values of n, the intensities of the spectra of orders $+r$ and $-r$ are equal, so that the spectra are symmetric with respect to the spectrum of order zero; in the case of even values of n the spectra are asymmetric with respect to the order zero, *i.e.*, the intensities of the orders $+r$ and $-r$ are not equal (excepted for $\delta = \pi/2$).

4. PROPERTIES OF THE INTENSITIES IN THE GENERAL CASE ($\rho \neq 0$)

We shall now show that Murty's conclusion for odd values of n will not hold in the general case.

Let us suppose that the intensities of the orders r and $-r$ should be equal, *i.e.*, $I_r = I_{-r}$, then we should have either $\phi_r = e^{i\gamma_r(\xi)}\phi_{-r}$ or $\phi_r = e^{i\gamma_r(\xi)}\phi_{-r}^*$.

(*a*) *First possibility*:

$$\phi_r = e^{i\gamma_r(\xi)}\phi_{-r}.$$

Introducing this substitution into the general equation (9) we obtain

$$2\frac{d\phi_{-r}}{d\xi} - [\phi_{-r+1}e^{i(\gamma_{r-1}-\gamma_r)} - \phi_{-r-1}e^{i(\gamma_{r+1}-\gamma_r)}] - a_n[\phi_{-r+n}e^{i(\gamma_{r-n}-\gamma_r)}e^{-i\delta} - \phi_{-r-n}e^{i(\gamma_{r+n}-\gamma_r)}e^{i\delta}] = \left(-2i\frac{d\gamma_r}{d\xi} + i\rho r^2\right)\phi_{-r}. \tag{18}$$

On the other hand, changing r into $-r$ in equation (9) gives

$$2\frac{d\phi_{-r}}{d\xi} - (\phi_{-r-1} - \phi_{-r+1}) - a_n(\phi_{-r-n}e^{-i\delta} - \phi_{-r+n}e^{i\delta}) = i\rho r^2\phi_{-r}. \tag{19}$$

Equations (18) and (19) must be identic, hence,

(1°) $d\gamma_r/d\xi = 0$ from which it follows that γ_r is a constant;

(2°) $e^{i(\gamma_{r-1}-\gamma_r)} = -1$ and $e^{i(\gamma_{r+1}-\gamma_r)} = -1$, from which

$$\gamma_r = r\pi; \tag{20}$$

(3°) $e^{i(\gamma_{r-n}-\gamma_r)}e^{-i\delta} = -e^{i\delta}$ and $e^{i(\gamma_{r+1}-\gamma_r)}e^{i\delta} = -e^{-i\delta}$,

from which taking into account (20),

$$\cos\delta = 0 \text{ or } \delta = (k + \tfrac{1}{2})\pi \text{ if } n \text{ is even},$$

and

$$\sin \delta = 0 \text{ or } \delta = k\pi \text{ when } n \text{ is odd.}$$

(*b*) *Second possibility*:

$$\phi_r = e^{i\gamma_r(\xi)}\phi_{-r}^*.$$

Making this substitution into the system (9) gives

$$2\frac{d\phi_{-r}^*}{d\xi} - [\phi^*_{-r+1}e^{i(\gamma_{r-1}-\gamma_r)} - \phi^*_{-r-1}e^{i(\gamma_{r+1}-\gamma_r)}]$$

$$- a_n[\phi^*_{-r+n}e^{-i\delta}e^{i(\gamma_{r-n}-\gamma_r)} - \phi^*_{-r-n}e^{i\delta}e^{i(\gamma_{r+n}-\gamma_r)}]$$

$$= i\left(\rho r^2 - 2\frac{d\gamma_r}{d\xi}\right)\phi^*_{-r}. \tag{21}$$

Changing r into $-r$ in the equation (9) and taking its complex conjugate, leads to

$$2\frac{d\phi_{-r}^*}{d\xi} - (\phi^*_{-r-1} - \phi^*_{-r+1}) - a_n(\phi^*_{-r-n}e^{i\delta} - \phi^*_{-r+n}e^{-i\delta})$$

$$= -i\rho r^2\phi_{-r}^*. \tag{22}$$

The identity of the second members of the equations (21) and (22) gives $d\gamma_r/d\xi = r^2\rho$ from which $\gamma_r = \rho r^2\xi + r\pi$ (when $\rho = 0$, then $\gamma_r = r\pi$). The first bracket in the left-hand side of equation (21) then reads $\phi^*_{-r-1}e^{i\rho\xi(1+2r)} - \phi^*_{-r+1}e^{i\rho\xi(1-2r)}$. Comparing this expression with the first bracket of the left-hand side of equation (12), it seems that their identity could never be realized. Hence, we may conclude *in the general case*, $\rho \neq 0$: *if* n *is even and* $\delta \neq (k + \frac{1}{2})\pi$ *or if* n *is odd and* $\delta \neq k\pi$ *the intensities of the orders* r *and* $-$ r *could never be equal.*

In the case $\rho = 0$, the same reasoning leads to the conclusion that the intensities of orders r and $-r$ cannot be equal if n is even. When n is odd, the second possibility predicts that the intensities of orders r and $-r$ could be equal and that the relation between ϕ_r and ϕ_{-r} should be $\phi_r = e^{i\pi r}\phi^*_{-r} = (-1)^r\phi_{-r}^*$. It is easy to verify that the solution (17) obeys this relation for odd values of n.

Ramachandra Rao's experimental observation[8] of a symmetric spectrum for $n = 3$ could be explained by the fact that his experimental conditions are so that $\rho \ll 1$ or that the phase angle $\delta = k\pi$.

5. Some General Considerations on the Series Solution of the General System (9)

Introducing the series $\phi_r = \sum_{k=0}^{\infty} A_{rk}\xi^k$ into the system (9), leads after careful inspection to the conclusion that the series must have the following form:

(*a*) $n = 2l + 1$, $m = 0, 1, 2, \ldots\ldots$

$$\left.\begin{aligned}
\phi_0 &= \sum_{k=0}^{\infty} A_{0k}\xi^k \\
\phi_{\pm(2l+1)\,m\pm1} &= \xi^{m+1}\sum_{k=0}^{\infty} A_{\pm(2l+1)\,m\pm1,k}\xi^k \\
\phi_{\pm(2l+1)\,m\pm2} &= \xi^{m+2}\sum_{k=0}^{\infty} A_{\pm(2l+1)\,m\pm2,k}\xi^k \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\phi_{\pm(2l+1)\,m\pm l} &= \xi^{m+l}\sum_{k=0}^{\infty} A_{\pm(2l+1)\,m\pm l,k}\xi^k \\
\phi_{\pm(2l+1)\,m\pm(l+1)} &= \xi^{m+l+1}\sum_{k=0}^{\infty} A_{\pm(2l+1)\,m\pm(l+1),k}\xi^k \\
\phi_{\pm(2l+1)\,m\pm(l+2)} &= \xi^{m+l}\sum_{k=0}^{\infty} A_{\pm(2l+1)\,m\pm(l+2),k}\xi^k \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\phi_{\pm(2l+1)\,m\pm(2l+1)} &= \xi^{m+1}\sum_{k=0}^{\infty} A_{\pm(2l+1)\,m\pm(2l+1),k}\xi^k
\end{aligned}\right\} \quad (23)$$

(*b*) $n = 2l$, $m = 0, 1, 2, \ldots\ldots$

$$\left.\begin{aligned}
\phi_0 &= \sum_{k=0}^{\infty} A_{0k}\xi^k \\
\phi_{\pm2lm\pm1} &= \xi^{m+1}\sum_{k=0}^{\infty} A_{\pm2lm\pm1,k}\xi^k \\
\phi_{\pm2lm\pm2} &= \xi^{m+2}\sum_{k=0}^{\infty} A_{\pm2lm\pm2,k}\xi^k \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\phi_{\pm2lm\pm l} &= \xi^{m+l}\sum_{k=0}^{\infty} A_{\pm2lm\pm l,k}\xi^k \\
\phi_{\pm2lm\pm(l+1)} &= \xi^{m+l}\sum_{k=0}^{\infty} A_{\pm2lm\pm(l+1),k}\xi^k \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\phi_{\pm2lm\pm2l} &= \xi^{m+1}\sum_{k=0}^{\infty} A_{\pm2lm\pm2l,k}\xi^k.
\end{aligned}\right\} \quad (24)$$

It may easily be verified that the lowest powers of ξ in those series are the same as those in the solution (17) of the system in the special case $\rho = 0$.

6. Explicit Calculation of the Intensities by the Series Method in the Case $n = 2$

According to the series expansions (24) we may write for $n = 2$,

$$\left.\begin{aligned} \phi_0 &= \sum_{k=0}^{\infty} A_{0k}\xi^k \\ \phi_{\pm 2m\pm 1} &= \frac{{a_2}^m \xi^{m+1}}{2^{m+1}m!} \sum_{k=0}^{\infty} A_{\pm 2m\pm 1,k}\xi^k \\ \phi_{\pm 2m\pm 2} &= \frac{{a_2}^{m+1} \xi^{m+1}}{2^{m+1}(m+1)!} \sum_{k=0}^{\infty} A_{\pm 2m\pm 2,k}\xi^k. \end{aligned}\right\} \tag{25}$$

The factors ${a_2}^m/2^{m+1}m!$ and ${a_2}^{m+1}/2^{m+1}(m+1)!$ are written for convenience and are chosen as the corresponding lowest power coefficients of the series expansion of the solution (17) for $\rho = 0$. Substituting the series (25) into the equation (9) and equating the coefficients of equal powers of ξ on both sides, we find the following relations between the coefficients A_{rk}:

$$\begin{aligned} &(m+k+2)A_{\pm 2m\pm 1,\,k+1} \mp A_{\pm 2m,\,k+1} \pm \frac{{a_2}}{2(m+1)} A_{\pm 2m\pm 2,k} \\ &\qquad \mp mA_{\pm 2m\mp 1,\,k+1}e^{\mp i\delta} \pm \frac{{a_2}^2}{4(m+1)} A_{\pm 2m\pm 3,k-1}e^{\pm i\delta} \\ &\quad = \frac{i\rho}{2}(2m+1)^2 A_{\pm 2m\pm 1,\,k} \qquad (m \geqslant 1) \end{aligned} \tag{26}$$

$$\begin{aligned} &(k+2)A_{\pm 1,\,k+1} \mp A_{0,\,k+1} \pm \frac{a_2}{2} A_{\pm 2,\,k} \mp \frac{a_2}{2} A_{\mp 1,\,k}e^{\mp i\delta} \\ &\qquad \pm \frac{{a_2}^2}{4} A_{\pm 3,\,k-1}e^{\pm i\delta} \\ &\quad = \frac{i\rho}{2} A_{\pm 1,\,k} \end{aligned} \tag{27}$$

$$\begin{aligned} &(m+k+2)A_{\pm 2m\pm 2,\,k+1} \mp \frac{m+1}{2a_2} A_{\pm 2m\pm 1,\,k} \pm \tfrac{1}{4} A_{\pm 2m\pm 3,\,k-1} \\ &\qquad \mp (m+1)A_{\pm 2m,\,k+1}e^{\mp i\delta} \pm \frac{{a_2}^2}{4(m+2)} A_{\pm 2m\pm 4,\,k-1}e^{\pm i\delta} \\ &\quad = \frac{i\rho}{2}(2m+2)^2 A_{\pm 2m\pm 2,\,k} \qquad (m \geqslant 0) \end{aligned} \tag{28}$$

$$(k+1)\,A_{0,k+1} - \tfrac{1}{4}A_{-1,k-1} + \tfrac{1}{4}A_{1,k-1} - \frac{{a_2}^2}{4}A_{-2,k-1}\,e^{-i\delta} + \frac{{a_2}^2}{4}A_{2,k-1}\,e^{i\delta} = 0. \tag{29}$$

It is seen that it is possible to go over from the relations with the upper signs to the relations with the lower signs by means of the substitution

$$\left.\begin{aligned} A_{rk} &\rightarrow (-1)^r A_{-r,k} \\ \delta &\rightarrow -\delta + \pi. \end{aligned}\right\} \tag{30}$$

Hence, if one has obtained the formula for a A_{rk}, the same transformation allows the immediate calculation of $A_{-r,k}$.

From the relations (26)–(29) we have calculated the general expressions for the first three coefficients. We refer to the appendix for an example of the detailed computation of one of the coefficients.

$$A_{2m+2,0} = e^{-i(m+1)\delta} \tag{31 a}$$

$$A_{2m+1,0} = e^{-im\delta} \tag{31 b}$$

$$A_{2m+2,1} = \frac{m+1}{4a_2}e^{-im\delta} + \frac{i\rho}{3}(m+1)(2m+3)\,e^{-i(m+1)\delta} \tag{31 c}$$

$$A_{2m+1,1} = -\frac{a_2}{2(m+1)}e^{-i(m+1)\delta} + \frac{i\rho}{12}(8m^2+12m+3)\,e^{-im\delta} + \frac{m}{12a_2}e^{-i(m-1)\delta} \tag{31 d}$$

$$\begin{aligned} A_{2m+2,2} = &-\tfrac{1}{4}e^{-i(m+1)\delta} - \frac{{a_2}^2}{4(m+2)}e^{-i(m+1)\delta} \\ &+ \frac{m(m+1)}{96{a_2}^2}e^{-i(m-1)\delta} \\ &+ \frac{i\rho}{24a_2}(m+1)(4m^2+10m+5)\,e^{-im\delta} \\ &- \frac{\rho^2}{90}(m+1)(2m+3)(2m+5)(5m+4)\,e^{-i(m+1)\delta} \end{aligned} \tag{31 e}$$

$$A_{2m+1,2} = -\tfrac{1}{8}\,e^{-im\delta} - \frac{{a_2}^2}{4(m+1)}\,e^{-im\delta} + \frac{m(m-1)}{480{a_2}^2}\,e^{-i(m-2)\delta}$$
$$- \frac{i\rho a_2}{24}\,\frac{8m^2+12m+7}{m+1}\,e^{-i(m+1)\delta}$$
$$+ \frac{i\rho}{144a_2}\,m(8m^2+12m+1)\,e^{-i(m-1)\delta}$$
$$- \frac{\rho^2}{360}(80m^4+304m^3+328m^2+113m+15)\,e^{-im\delta}. \qquad (31f)$$

From those formulæ we may obtain the expressions for the intensities of the spectra, as far as the third term included of the series expansion. The final formulæ are:

$$I_{\pm 2m\pm 1} = \frac{\xi^{2m+2}{a_2}^{2m}}{2^{2m+2}(m!)^2}\left\{1 \pm \xi\left(-\frac{a_2}{m+1}+\frac{m}{6a_2}\right)\cos\delta\right.$$
$$+ \xi^2\left[-\frac{1}{4}-\frac{{a_2}^2}{4}\,\frac{2m+1}{(m+1)^2}+\frac{m^2}{144{a_2}^2}\right.$$
$$- \frac{m}{12(m+1)}\cos 2\delta + \frac{m(m-1)}{240\,{a_2}^2}\cos 2\delta$$
$$+ \frac{\rho}{3}\left(-\frac{a_2}{m+1}+\frac{m}{12a_2}\right)\sin\delta$$
$$\left.\left. - \frac{\rho^2}{720}(256m^3+352m^2+92m+15)\right] + \ldots\ldots\right\}. \qquad (32)$$

$$I_{\pm 2m\pm 2} = \frac{\xi^{2m+2}{a_2}^{2m+2}}{2^{2m+2}[(m+1)!]^2}\left\{1 \pm \xi\,\frac{m+1}{2a_2}\cos\delta\right.$$
$$+ \xi^2\left[-\frac{1}{2}-\frac{{a_2}^2}{2(m+2)}+\frac{(m+1)^2}{16{a_2}^2}\right.$$
$$+ \frac{m(m+1)}{48{a_2}^2}\cos 2\delta + \frac{\rho}{12a_2}(m+1)\sin\delta$$
$$\left.\left. - \frac{\rho^2}{45}(m+1)(2m+3)(8m+5)\right] + \ldots\ldots\right\}. \qquad (33)$$

Those series show an asymmetry in the spectra with regard to the zero order. The asymmetry is contained in the second term and is not affected by the introduction of terms in ρ; hence, there is consistency with the results

of the preliminary theory, in which for $\rho = 0$ and even values of n, an asymmetric spectrum is obtained. It is also seen that the symmetry is re-established when the phase angle $\delta = (k + \frac{1}{2})\pi$, confirming the result of Murty and the conclusion of Section 4.

Although the series (32) and (33) are cut off after the third terms, limiting their practical use to values of $\xi < 1$, it is possible to obtain some properties from them, which agree qualitatively with the experiment. For $m = 0$ they show that the intensity of the order -1 is stronger than the intensity of the order $+1$, and that the intensity of the order $+2$ is stronger than the intensity of the order -2, facts that are confirmed qualitatively by the experiments of Ramachandra Rao.[8]

Taking for instance $a_2 = 1$, the same features may be established for $m = 1$; for $m = 2$, however, $I_5 = I_{-5}$ (in the considered approximation) and from then on all the intensities of positive orders should be stronger than the intensities of corresponding negative orders.

It has already been remarked by Ramachandra Rao that the nature of the asymmetry in the case of parallel sound waves for even values of n is entirely different from the asymmetry caused by the oblique incidence of the light, in which the intensity of any positive order is greater than the intensity of the corresponding negative order.

7. Explicit Calculation of the Intensities by the Series Method in the Case $n = 3$

Taking into account the form of the series (23), we put for $n = 3$,

$$\left.\begin{aligned}
\phi_0 &= \sum_{k=0}^{\infty} A_{0k}\xi^k \\
\phi_{\pm 3m\pm 1} &= \frac{\xi^{m+1}a_3{}^m}{2^{m+1}m!}\sum_{k=0}^{\infty} A_{\pm 3m\pm 1,\,k}\xi^k \\
\phi_{\pm 3m\pm 2} &= \frac{\xi^{m+2}a_3{}^{m+1}}{2^{m+2}(m+1)!}\sum_{k=0}^{\infty} A_{\pm 3m\pm 2,\,k}\xi^k \\
\phi_{\pm 3m\pm 3} &= \frac{\xi^{m+1}a_3{}^{m+1}}{2^{m+1}(m+1)!}\sum_{k=0}^{\infty} A_{\pm 3m\pm 3,\,k}\xi^k
\end{aligned}\right\} \qquad (34)$$

The factors preceding the series have again been chosen as the corresponding lowest power coefficients of the series expansion of the solution (17) in the case $\rho = 0$, excepted for the factor of $\phi_{\pm 3m\pm 2}$, which should read

$$\left(\frac{m+1}{a_3} - 2e^{\mp i\delta}\right)\frac{\xi^{m+2}a_3{}^{m+1}}{2^{m+3}(m+1)!},$$

but which has been chosen in the above form in order to simplify the relations between the series coefficients. Substitution of the series (34) into the equation (9) leads to the following relations between the coefficients A_{rk},

$$(m+k+2)\,A_{\pm 3m\pm 1,\,k+1} \mp A_{\pm 3m,\,k+1} \pm \frac{\alpha_3}{4\,(m+1)}\,A_{\pm 3m\pm 2,\,k-1}$$
$$\mp mA_{\pm 3m\mp 2,\,k+1}e^{\mp i\delta} \pm \frac{{\alpha_3}^2}{4\,(m+1)}\,A_{\pm 3m\pm 4,\,k-1}e^{\pm i\delta}$$
$$= \frac{i\rho}{2}\,(3m+1)^2\,A_{\pm 3m\pm 1,\,k} \qquad (m \geqslant 1) \tag{35}$$

$$(k+2)\,A_{\pm 1,\,k+1} \mp A_{0,\,k+1} \pm \frac{\alpha_3}{4}\,A_{\pm 2,\,k-1} \mp \frac{{\alpha_3}^3}{4}\,A_{\mp 2,\,k-1}e^{\mp i\delta}$$
$$\pm \frac{{\alpha_3}^2}{4}\,A_{\pm 4,\,k-1}e^{\pm i\delta}$$
$$= \frac{i\rho}{2}\,A_{\pm 1,\,k} \tag{36}$$

$$(m+k+3)\,A_{\pm 3m\pm 2,\,k+1} \mp \frac{m+1}{\alpha_3}\,A_{\pm 3m\pm 1,\,k+1} \pm A_{\pm 3m\pm 3,\,k+1}$$
$$\mp (m+1)\,A_{\pm 3m\mp 1}e^{\mp i\delta} \pm \frac{{\alpha_3}^2}{4\,(m+2)}\,A_{\pm 3m\pm 5,\,k-1}e^{\pm i\delta}$$
$$= \frac{i\rho}{2}\,(3m+2)^2\,A_{\pm 3m\pm 2,\,k} \qquad (m \geqslant 1) \tag{37}$$

$$(k+3)\,A_{\pm 2,\,k+1} \mp \frac{1}{\alpha_3}\,A_{\pm 1,\,k+1} \pm A_{\pm 3,\,k+1} \mp A_{\mp 1,\,k+1}e^{\mp i\delta}$$
$$\pm \frac{{\alpha_3}^2}{8}\,A_{\pm 5,\,k-1}e^{\pm i\delta}$$
$$= 2i\rho A_{\pm 2,\,k} \tag{38}$$

$$(m+k+2)\,A_{\pm 3m\pm 3,\,k+1} \mp \tfrac{1}{4}\,A_{\pm 3m\pm 2,\,k-1} \pm \tfrac{1}{4}\,A_{\pm 3m\pm 4,\,k-1}$$
$$\mp (m+1)\,A_{\pm 3m,\,k+1}e^{\mp i\delta} \pm \frac{{\alpha_3}^2}{4\,(m+2)}\,A_{\pm 3m\pm 6,\,k-1}e^{\pm i\delta}$$
$$= \frac{i\rho}{2}\,(3m+3)^2\,A_{\pm 3m\pm 3,\,k} \qquad (m \geqslant 0) \tag{39}$$

$$(k+1)\,A_{0,\,k+1} - \tfrac{1}{4}\,A_{-1,\,k-1} + \tfrac{1}{4}\,A_{1,\,k-1}$$
$$- \frac{{\alpha_3}^2}{4}\,A_{-3,\,k-1}e^{-i\delta} + \frac{{\alpha_3}^2}{4}\,A_{3,\,k-1}e^{i\delta} = 0. \tag{40}$$

In order to transform the formulæ with the upper signs into the formulæ with the lower signs the substitution:

$$\left.\begin{array}{lcl} A_{rk} & \rightarrow & (-1)^r A_{-r,k} \\ \delta & \rightarrow & -\delta \end{array}\right\} \tag{41}$$

must be performed. The same transformation changes the formulæ for A_{rk} into the expressions for $A_{-r,k}$.

From the relations (35)–(40) one obtains,

$$A_{3m+3,0} = e^{-i(m+1)\delta} \tag{42 a}$$

$$A_{3m+1,0} = e^{-im\delta} \tag{42 b}$$

$$A_{3m+2,0} = \left(\frac{m+1}{2a_3} - e^{-i\delta}\right) e^{-im\delta} \tag{42 c}$$

$$A_{3m+3,1} = \tfrac{3}{4} i\rho (m+1)(2m+3)\, e^{-i(m+1)\delta} \tag{42 d}$$

$$A_{3m+1,1} = \frac{i\rho}{4}(m+1)(6m+1)\, e^{-im\delta} \tag{42 e}$$

$$A_{3m+2,1} = \frac{i\rho}{24a_3}(m+1)(18m^2+33m+10)\, e^{-im\delta} - \frac{i\rho}{4}(6m^2+11m+6)\, e^{-i(m+1)\delta} \tag{42 f}$$

$$A_{3m+3,2} = \frac{m+1}{24a_3} e^{-im\delta} - \frac{1}{4} e^{-i(m+1)\delta} - \frac{a_3^2}{4(m+2)} e^{-i(m+1)\delta} - \frac{9}{160}\rho^2 (m+1)(2m+3)(2m+5) \times (5m+4)\, e^{-i(m+1)\delta} \tag{42 g}$$

$$A_{3m+1,2} = \frac{m}{96a_3} e^{-i(m-1)\delta} - \frac{1}{8} e^{-im\delta} + \frac{a_3}{8(m+1)} e^{-i(m+1)\delta} - \frac{a_3^2}{4(m+1)} e^{-im\delta} - \frac{\rho^2}{480}(540m^4 + 1692m^3 + 1305m^2 + 213m + 20)\, e^{-im\delta} \tag{42 h}$$

$$A_{3m+3,3} = \frac{i\rho}{96a_3}(m+1)(6m^2+15m+7)\, e^{-im\delta} + \text{terms in } i\rho\, e^{-i(m+1)\delta} \tag{42 i}$$

$$A_{3m+1,3} = \frac{i\rho a_3}{96}\frac{18m^2+21m+13}{m+1}\,e^{-i(m+1)\delta} + \frac{i\rho}{384a_3}\,m(6m^2+7m-1)\,e^{-i(m-1)\delta} + \text{terms in } i\rho\, e^{-im\delta}. \tag{42j}$$

In (42 *i*) and (42 *j*) only those terms are written that give a contribution to the term in ξ^3 in the series for the intensities.

We have finally calculated the expressions for the intensities as far the first term of the series contributing to the asymmetry. We find

$$I_{\pm 3m\pm 1} = \frac{\xi^{2m+2}a_3{}^{2m}}{2^{2m+2}(m!)^2}\left\{1+\xi^2\left[-\frac{1}{4}-\frac{a_3{}^2}{2(m+1)} + \frac{1}{4}\left(\frac{m}{12a_3}+\frac{a_3}{m+1}\right)\cos\delta - \frac{\rho^2}{240}(432m^3+390m^2+3m+5)\right] \pm \xi^3\frac{\rho}{24}\left(\frac{m}{4a_3}+\frac{5a_3}{m+1}\right)\sin\delta+\ldots.\right\} \tag{43}$$

$$I_{\pm 3m\pm 2} = \frac{\xi^{2m+4}a_3{}^{2m+2}}{2^{2m+4}[(m+1)!]^2}\left\{1+\frac{(m+1)^2}{4a_3{}^2}-\frac{m+1}{a_3}\cos\delta \mp \xi\frac{2}{3}\rho\frac{m+1}{a_3}\sin\delta+\ldots.\right\} \tag{44}$$

$$I_{\pm 3m\pm 3} = \frac{\xi^{2m+2}a_3{}^{2m+2}}{2^{2m+2}[(m+1)!]^2}\left\{1-\frac{\xi^2}{2}\left[1+\frac{a_3{}^2}{m+2} - \frac{m+1}{6a_3}\cos\delta+\frac{9}{40}\rho^2(m+1)(2m+3) \times(8m+5)\right] \pm \xi^3\frac{\rho}{24a_3}(m+1)\sin\delta+\ldots.\right\}. \tag{45}$$

The intensities of those spectra also show an asymmetry with respect to the zero order line. The asymmetry disappears when ρ is put equal to zero, thus confirming the results of the preliminary theory, or when the phase angle $\delta = k\pi$ in accordance with the conclusions of Section 4.

From the formulæ it is to be expected that at least for values of $\xi < 1$, the intensities of the orders $3m+1$ and $3m+3$ should respectively be

greater than the intensities of the orders $-3m-1$ and $-3m-3$, and that the intensity of the order $-3m-2$ should be stronger than the intensity of the order $3m+2$. Here also we have a marked difference with the nature of the asymmetry in the case of oblique incidence of the light.

With regard to the lowest power term in ξ we may also notice that for small values of ξ the intensities of orders $\pm 3m \pm 2$ should be much weaker than the intensities of the orders $\pm 3m \pm 1$ and $\pm 3m \pm 3$.

We wish to express our sincere thanks to Prof. Dr. M. Nuyens for his continuous interest in our work.

Summary

The theory of the diffraction of light by two superposed parallel supersonic waves, consisting of the fundamental tone and the n-th harmonic is developed, starting from the wave equation for the electric field of the light. The Fourier series method, first used in Raman and Nath's generalized theory, is here employed to derive a system of difference-differential equations for the amplitudes of the diffracted light waves. The r-th order spectrum makes an angle $\theta_r = -Arc \sin r\lambda/\lambda^*$ with the direction of the incident light and presents a change of frequency $-r\nu^*$. In the case that the right-hand side of the difference-differential equations may be neglected, the exact solution is obtained by means of a complex function method. From the structure of the general system of difference-differential equations it is shown that the intensities of the orders r and $-r$ are always different for an even as well as for an odd ratio of the sound frequencies, excepted for some special values of the phase angle of the sound waves. A solution of the general system for $n = 2$ is written in the form of a power series in ξ, the terms of which are calculated till the third ones included; the asymmetry of the intensities of opposite orders is not due to the terms in ρ. In the case $n = 3$ a series solution also leads to an asymmetric pattern with respect to the zero order line; the terms in ρ are here responsible for the asymmetry, so that the symmetry property reappears for $\rho = 0$, in accordance with the simplified theory based on Raman and Nath's preliminary theory.

References

1. Bergmann, L. and Fues, E. *Z. Physik*, 1938, **109**, 1.
2. Nagendra Nath, N. S. .. *Proc. Cambr. Phil. Soc.*, 1938, **34**, 213.
3. Nagabhushana Rao, K. .. *Proc. Ind. Acad. Sci.*, 1938, **8 A**, 124.
4. Raman, C. V. and Nagendra Nath, N. S. *Ibid.*, 1935, **2 A**, 406.

5. Bergmann, L. .. *Z. Hochfrequenztech.*, 1934, **43**, 83.
6. Hiedemann, E. and Hoesch, K. H. *Z. Physik.*, 1935, **98**, 141.
7. Govinda Rao, M. A. .. *Proc. Ind. Acad. Sci.*, 1938, **8 A**, 6.
8. Ramachandra Rao, B. .. *Ibid.*, 1949, **29 A**, 16.
9. Murty, J. S. .. *J. Acoust. Soc. Am.*, 1954, **26**, 970.
10. Zankel, K. L. and Hiedemann, E. A. *Naturwissensch.*, 1958, **45**, 157.
11. Zankel, K. L. .. Privhte comunication.
12. Raman, C. V. and Nagendra Nath, N. S. *Proc. Ind. Acad. Sci.*, 1936, **3 A**, 119.
13. Watson, G. N. .. *Treatise on the Theory of Bessel Functions*, Cambridge, 1944, p. 14.

APPENDIX

Example of the calculation of a series coefficient from the relations existing between them:

Firstly we must remark that the boundary condition $\phi_0(0) = 0$ always requires $A_{00} = 1$. The second boundary condition $\phi_r(0) = 0\,(r \neq 0)$ is always satisfied, a power of ξ greater than zero always preceding the series as a factor.

We shall now give a typical example of the calculation of a coefficient in the series ϕ_{2m+2}, namely $A_{2m+2,1}$. The principle is quite the same for the establishment of the other coefficients, those for $n = 3$ included. Only the length of the computations increases considerably as one tries to obtain higher coefficients.

Putting into the relation (28) $k = 0$, observing that the coefficients, the second indices of which are negative, should be identically zero, and taking into account the values of $A_{2m+1,0}$ and $A_{2m+2,0}$ (31 *a*, *b*), previously calculated by the same method, one has

$$(m+2)\,A_{2m+2,1} - (m+1)\,A_{2m,1}\,e^{-i\delta} = \frac{m+1}{2a_2}\,e^{-im\delta} + 2\,(m+1)^2\,i\rho\,e^{-i\,(m+1)\,\delta}.$$

We write this relation for $m = 0,\ 1,\ 2,\ \ldots\ldots,\ m-1,\ m$:

$$2A_{21} - 1A_{01}\,e^{-i\delta} = \frac{1}{2a_2} + 2.1^2\,i\rho\,e^{-i\delta}$$

$$3A_{41} - 2A_{21}\,e^{-i\delta} = \frac{2}{2a_2}\,e^{-i\delta} + 2.2^2\,i\rho\,e^{-i2\delta}$$

$$4A_{61} - 3A_{41}\,e^{-i\delta} = \frac{3}{2a_2}\,e^{-i2\delta} + 2.3^2\,i\rho\,e^{-i3\delta}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(m+1)\,A_{2m,1} - mA_{2m-2,1}\,e^{-i\delta} = \frac{m}{2a_2}\,e^{-i\,(m-1)\,\delta} + 2.m^2 i\rho\,e^{-im\delta}$$

$$(m+2)\,A_{2m+2,1} - (m+1)\,A_{2m,1}\,e^{-i\delta} = \frac{m+1}{2a_2}\,e^{-im\delta} + 2\,(m+1)^2 i\rho\,e^{-i\,(m+1)\,\delta}.$$

From (29) we get for $k = 0$, $A_{01} = 0$. Multiplying both sides of the above equations respectively by $e^{-im\delta}$, $e^{-i(m-1)\delta}$, $e^{-i(m-2)\delta}, \ldots, e^{-i\delta}$ and 1, and adding them, we obtain

$$(m+2)\, A_{2m+2,\,1} = \frac{1}{2\alpha_2} \sum_{p=1}^{m+1} p\, e^{-im\delta} + 2i\rho \sum_{p=1}^{m+1} p^2 e^{-i(m+1)\delta}.$$

Using the formulæ

$$\sum_{p=1}^{n} p = \frac{n(n+1)}{2}, \quad \sum_{p=1}^{n} p^2 = \frac{n(n+1)(2n+1)}{6}$$

there comes finally

$$A_{2m+2,\,1} = \frac{m+1}{4_{\alpha 2}} e^{-im\delta} + \frac{(m+1)(2m+3)}{3} i\rho\, e^{-i(m+1)\delta}.$$

It is perhaps interesting to remark, that if one wishes to calculate a great number of series coefficients one may first calculate $A_{00}, A_{10}, \ldots,$ $A_{2p+1,\,0}$ (p being a given number), then calculate $A_{01}, A_{11}, \ldots, A_{2p,\,1}$, further compute $A_{02}, A_{12}, \ldots, A_{2p-1,\,2}$, etc. Hereby the relations (26) and (28) are used alternatively.

Analogue methods may be used for the direct calculation of the coefficients in the case $n = 3$. This method is particularly suited, for given values of the parameter ρ and the phase angle δ, when using a computing machine.

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